JOHANNES SIEMONS

ON A CLASS OF PARTIALLY ORDERED SETS AND THEIR LINEAR INVARIANTS

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ABSTRACT. Let $(\mathscr{L}, <)$ be a finite partially ordered set with rank function. Then $\mathscr{L}$ is the disjoint union of the classes $\mathscr{L}_k$ of elements of rank k and the order relation between elements in $\mathscr{L}_k$ and $\mathscr{L}_{k+1}$ can be represented by a matrix S_k. We study partially ordered sets which satisfy linear recurrence relations of the type $S_k(S_k^T) - c_k(S_{k-1})^T S_{k-1} = (d_k^+ - c_k d_k^-)\mathrm{Id}$ for all k and certain coefficients d_{k^+}, d_{k^-} and c_k.

1. Introduction

Let $(\mathscr{L}, <)$ be a finite partially ordered set with unique minimal element 0. We suppose that $\mathscr{L}$ admits a rank function $|\ |: \mathscr{L} \to \mathbb{N}$ so that for every x in $\mathscr{L}$ any saturated chain $0 < x_1 < \cdots < x_r = x$ has length $r = |x|$ depending only on x. We set $\mathscr{L}_k = \{x \mid x \in \mathscr{L}, |x| = k\}$ and $n_k = |\mathscr{L}_k|$. We are interested in the following regularity conditions:

RI *For every $k = 0, 1, \ldots$ and every x in $\mathscr{L}_k$ the numbers $d_k^+ := |\{y \mid y \in \mathscr{L}_{k+1}, y > x\}|$ and $d_k^- := |\{z \mid z \in \mathscr{L}_{k-1}, x > z\}|$ depend only on k.*

RII *For every $k = 1, 2, \ldots$ there is a constant c_k such that if $x' \neq x$ belong to $\mathscr{L}_k$ then $|\{y \mid y \in \mathscr{L}_{k+1}, x' < y > x\}| = c_k |\{z \mid z \in \mathscr{L}_{k-1}, x' > z < x\}|$.*

Such a partially ordered set will be called *join–meet regular.* The main examples which motivate this definition are given in Section 2.

A poset with rank function can be represented by a family of incidence matrices $S_0, S_1, S_2, \ldots, S_k, \ldots$ where the rows of S_k are indexed by the elements in $\mathscr{L}_k$, the columns indexed by the elements of $\mathscr{L}_{k+1}$ and where $(S_k)_{x,y} = 1$ if $x < y$ and $(S_k)_{x,y} = 0$ otherwise. In this paper we shall determine some of the linear invariants of these matrices over various fields.

The first result gives the relationship between local parameters such as d_k^+, d_k^-, c_k and the size of $\mathscr{L}_k$. It is a consequence of a basic lemma which translates the regularity conditions into a simple matrix equation. From this we also obtain the eigenvalue distribution of $S_k \cdot S_k^T$ and hence the rank of the incidence matrices.

In Section 3 we go on to consider tactical decompositions. Here two

Geometriae Dedicata **41:** 219–228, 1992.

families of incidence matrices arise; their invariants are treated in a very similar way.

Section 4 deals with *symmetric* tactical decompositions in which the sets $\mathscr{L}_k$ and $\mathscr{L}_{k+1}$ are decomposed into equally many classes for some values of k. The main result shows that the rational congruence relations which give rise to the Bruck–Ryser theorem on symmetric 2-designs remain in force in this much more general situation.

2. Incidence matrices and eigenvalues

Among the principal examples of join–meet regular posets are finite dimensional projective spaces over a finite field. We consider these as the lattice of subspaces naturally ordered by inclusion; the regularity condition II essentially is the Grassman identity.

Similarly, the 'characteristic 1' analogue of the above, namely the boolean algebra of subsets of a finite set, is join–meet regular.

There are many other examples. For instance, 2-designs can be viewed in this way: Let $\mathscr{P}$ and $\mathscr{B}$ be the point and the block sets of a 2-(v, k, μ) design. On $\mathscr{L} = \{0\} \cup \mathscr{P} \cup \mathscr{B}$ define an order by $0 < \not{p} < \not{b}$ if and only if $\not{p}$ is on $\not{b}$. Then $\mathscr{L}$ is join–meet regular with parameters $d_0^+ = v$, $d_2^- = k$, $c_1 = \mu$ and $c_2 = 0$. And conversely, a join–meet regular set of this type is a 2-design.

There are also general constructions. We mention a few:

1. The truncation – retaining the lower part – of a join–meet regular poset is join–meet regular.
2. Mirrors: if $\mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2 \cup \cdots \cup \mathscr{L}_k \cup \mathscr{L}_{k+1} \cup \cdots$ is join–meet regular we select some k and form a *mirrored version* $\mathscr{L}^* = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2 \cup \cdots \cup \mathscr{L}_{k-1} \cup \mathscr{L}_k \cup \mathscr{L}_{k-1} \cdots \mathscr{L}_2 \cup \mathscr{L}_1 \cup \mathscr{L}_0$ in which the order is reversed in the upper part.
3. Co-catenation: if $\mathscr{L}$ has a unique maximal element $1_{\mathscr{L}}$ and if $\mathscr{M}$ is join–meet regular with minimal element $0_{\mathscr{M}}$ then $\mathscr{L} \cup \mathscr{M}$ is join–meet regular upon identifying $1_{\mathscr{L}} = 0_{\mathscr{M}}$.
4. Direct or tensor products of join–meet regular sets retain the property.

Let now $\mathscr{L}$ be a partially ordered set with rank function. Every pair $k \neq m$ of integers gives rise to an incidence structure $\mathscr{L}_{k,m} = (\mathscr{L}_k, \mathscr{L}_m; \mathrm{I})$ in which incidence is induced by the order relation. As above we denote the incidence matrix of $\mathscr{L}_{k,k+1}$ by S_k and identify a standard basis vector of $\mathbb{R}^{n_k}$ with the corresponding element of $\mathscr{L}_k$.

Suppose that $\mathscr{L}$ satisfies the regularity condition RI. Then $S_k(S_k)^{\mathrm{T}}x = S_k(\Sigma_{x<y}y)$ with y in $\mathscr{L}_{k+1}$ and further $S_k(\Sigma_{x<y}y) = d_k^+x + \Sigma_{x \neq x'} f(x')x'$ where

$f(x')$ is the number of y for which $x < y > x'$. Similarly, we get $(S_{k-1})^{\mathrm{T}} S_{k-1} x = d_k^- x + \Sigma_{x \neq x'} f^*(x') x'$ where $f^*(x')$ now is the number of z in $\mathscr{L}_{k-1}$ for which $x > z < x'$. Hence we have the basic

LEMMA 2.1. *If* $(\mathscr{L}, <)$ *satisfies* RI *then* RII *holds if and only if there are constants* c_k *for every* k *in* $\{|x| \mid x \in \mathscr{L}\}$ *such that* $S_k(S_k)^{\mathrm{T}} - c_k(S_{k-1})^{\mathrm{T}} S_{k-1} = (d_k^+ - c_k d_k^-)\mathrm{Id}$.

THEOREM 2.2. *Let* $(\mathscr{L}, <)$ *be join–meet regular. Then for each value of k in* $\{|x| \mid x \in \mathscr{L}\}$:

(i) $n_{k-1} < n_k$ *implies* $d_k^+ \geqslant c_k d_k^-$, *and*

(ii) $d_k^+ > c_k d_k^-$ *implies* $n_k \leqslant n_{k+1}$.

This theorem is reminiscent of Fisher's inequality. It suggests that particular attention should be paid to the case $n_{k-1} = n_k$; we shall come back to this question in Section 4.

Proof. In Lemma 2.1 all matrices are symmetric and non-negative definite and have size n_k. Hence all their eigenvalues are non-negative. The result will follow from an inspection of the lowest eigenvalues. If $n_{k-1} < n_k$ then $(S_{k-1})^{\mathrm{T}} S_{k-1}$ is singular so that $d_k^+ - c_k d_k^-$ is the least eigenvalue of $S_k(S_k)^{\mathrm{T}}$ and hence is non-negative. Conversely, if the coefficient on the right-hand side in Lemma 2.1 is positive then $S_k(S_k)^{\mathrm{T}}$ is non-singular and so in particular $n_{k+1} \geqslant n_k$. □

If $\mathscr{S} = (\mathscr{P}, \mathscr{B}; \mathrm{I})$ is a finite incidence structure in general, with 'point set' $\mathscr{P}$, 'block set' $\mathscr{B}$, incidence relation $\mathrm{I} \leqslant \mathscr{P} \times \mathscr{B}$ and incidence matrix S, the *spectrum* of $\mathscr{S}$ is the collection of eigenvalues of the matrix SS^{T}. This will be denoted by spec($\mathscr{S}$) and we take account of the multiplicities of eigenvalues. Clearly the sum of the multiplicities is $|\mathscr{P}|$. Note that the definition is slightly different from the usual one in the case of graphs where the eigenvalues of the adjacency matrix $A = SS^{\mathrm{T}} - D$ are considered; here D is the diagonal matrix of vertex degrees.

In particular, we can speak about the spectrum of $\mathscr{L}_{k,k+1}$ $(k = 0, 1, \ldots)$ for any poset with rank function. In the case of join–meet regular posets spectra can be determined completely. First we need

LEMMA 2.3. *Let* $\alpha \colon V \to V^*$ *and* $\beta \colon V^* \to V$ *be linear maps between vector spaces over some field. Denote by* E_λ *rhe eigenspace of the map* $\beta\alpha \colon V \to V$ *and by* E_λ^* *the eigenspace of the map* $\alpha\beta \colon V^* \to V^*$ *for the same value* λ *in both cases. If* $\lambda \neq 0$ *then* $\alpha \colon E_\lambda \to E_\lambda^*$ and $\beta \colon E_\lambda^* \to E_\lambda$ *are isomorphisms.*

Proof. For x in E_λ we have $\alpha\beta(\alpha x) = \alpha(\beta\alpha x) = \lambda\alpha x$ so that $\alpha \colon E_\lambda \to E_\lambda^*$, even injectively as long as $\lambda \neq 0$. Apply the same argument to β. □

THEOREM 2.4. *Let* $(\mathscr{L}, <)$ *be join–meet regular and suppose that we have* $n_0 \leqslant n_1 \leqslant \cdots \leqslant n_k$ *up to some value of* k. *If* $\lambda_{0,k} > \lambda_{1,k} > , \ldots, > \lambda_{t,k}$ *denote the distinct values in* $\mathrm{spec}(\mathscr{L}_{k,k+1})$, *with corresponding multiplicities* $\mu_{0,k}, \mu_{1,k}, \ldots, \mu_{t,k}$, *then the* $\lambda_{i,k}$ *are integers and:*

(i) $t = k$ *and* $\mu_{i,k} = n_i - n_{i-1}$ *for* $0 \leqslant i \leqslant k$,

(ii) $\lambda_{0,k} = d_k^+ + \Sigma_{0 \leqslant s < k}(\Pi_{s<j\leqslant k} c_j)(d_s^+ - d_{s+1}^-)$ and
$\lambda_{i,k} = d_k^+ + \Sigma_{i \leqslant s < k}(\Pi_{s<j\leqslant k} c_j)(d_s^+ - d_{s+1}^-) - (\Pi_{i\leqslant j\leqslant k} c_j) d_i^-$
for $0 < i \leqslant k$. *In particular,* $\lambda_{t,k} = d_k^+ - c_k d_k^-$.

Proof. When $\mathscr{S} = (\mathscr{P}, \mathscr{B}; \mathrm{I})$ is an incidence structure, its dual is the incidence structure $\mathscr{S}' = (\mathscr{B}, \mathscr{P}; \mathrm{I}')$ in which the role of points and blocks is interchanged while incidence is as in $\mathscr{S}$. Thus, if $\mathscr{S}$ has incidence matrix S, then $\mathscr{S}'$ has incidence matrix S^{T}. It follows therefore from Lemma 2.3 that $\mathrm{spec}(\mathscr{S}')$ is obtained by *appending* $|\mathscr{B}| - |\mathscr{P}|$ zeros to $\mathrm{spec}(\mathscr{S})$ (or *deleting* $|\mathscr{P}| - |\mathscr{B}|$ zeros, if $|\mathscr{B}| < |\mathscr{P}|$).

This applies in particular to the incidence structures $\mathscr{L}_{k,k+1}$. Hence if $\lambda_{0,k-1} > \lambda_{1,k-1} > , \ldots, > \lambda_{t^*,k-1}$ are the special values of $\mathscr{L}_{k-1,k}$ then those of $\mathscr{L}_{k,k-1}$ are $\lambda_{0,k-1} > \lambda_{1,k-1} >, \ldots, > \lambda_{t^*,k-1} \geqslant \lambda_{t^*+1,k-1}$ where $\lambda_{t^*+1,k-1} = 0$ has multiplicity $n_k - n_{k-1}$. It follows from Lemma 2.1 that the values in $\mathrm{spec}(\mathscr{L}_{k,k+1})$ are

$$c_k\lambda_{0,k-1} + (d_k^+ - c_k d_k^-) > c_k\lambda_{1,k-1} + (d_k^+ - c_k d_k^-) >, \ldots,$$
$$> c_k\lambda_{t^*,k-1} + (d_k^+ - c_k d_k^-) \geqslant (d_k^+ - c_k d_k^-).$$

The formulae under (i) and (ii) are obtained by induction on k. It is now clear that the $\lambda_{i,k}$ are rational numbers. However, as eigenvalues of an integer matrix, they are algebraic integers and hence ordinary integers. □

REMARKS. 1. It is clear from the proof that the assumption $n_k \leqslant n_{k+1}$ is not really essential; the general case can be treated similarly.

2. In characteristic zero the matrices S and SS^{T} have the same rank. (In characteristic $p > 0$ this may *not* be the case!) Therefore the theorem gives the rank of S_k for all values of k. In particular, $\mathrm{rank}(S_k) = n_k$ unless $(d_k^+ - c_k d_k^-) = 0$, by Theorem 2.2. About this question see also Section 9 in Stanley [10].

3. Defining the *diameter* of an incidence structure as the usual diameter in the bipartite incidence graph, one can show that the diameter is bounded by the number of distinct eigenvalues in $\mathrm{spec}(\mathscr{S})$. It follows therefore that $\mathscr{L}_{k,k+1}$ has diameter at most $k + 1$.

4. If G is the automorphism group of $\mathscr{L}$, its action on each $\mathscr{L}_k$ naturally extends to a linear action on $\mathbb{R}^{n_k}$. The incidence matrices give rise to maps

$S_k S_k^{\mathrm{T}}: \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ and it is clear that these are G-maps. Therefore each eigenspace E_λ for λ in $\operatorname{spec}(\mathscr{L}_{k,k+1})$ is G-invariant so that we obtain decompositions of $\mathbb{R}^{n_k} \geqslant \mathbb{R}^{n_{k-1}} \geqslant \cdots \geqslant \mathbb{R}^{n_0}$ into invariant G-modules. See Lemma 2.3 and also [4]. In the case of the subspace or subset lattices mentioned at the beginning of this section, the E_λ's are in fact irreducible.

2.1. *The Modular Case*

While we have seen that our spectra are integer valued, we have so far considered these only over the real numbers. One is tempted, therefore, to make at least some remarks in the case of a field in non-zero characteristic. The first obstacle is the fact that a symmetric matrix over such a field may not decompose and hence may not be diagonizable. We shall therefore say that the spectrum of $\mathscr{L}_{k,k+1}$ *exists over* F, denoted by $\operatorname{spec}_F(\mathscr{L}_{k,k+1})$, if $S_k S_k^{\mathrm{T}}$ is diagonizable as a matrix over F. A partial result is

PROPOSITION 2.5. *Let* $(\mathscr{L}, <)$ *be a join–meet regular partially ordered set and let* F *be a field. Suppose that* $\operatorname{spec}_F(\mathscr{L}_{k-1,k})$ *exists for some value of* k. *Then* $\operatorname{spec}_F(\mathscr{L}_{k,k+1})$ *exists unless* 0_F *belongs to* $\operatorname{spec}_F(\mathscr{L}_{k-1,k})$.

Proof. Recall the formula $S_k(S_k)^{\mathrm{T}} = c_k(S_{k-1})^{\mathrm{T}} S_{k-1}^{+}(d_k^{+} - c_k d_k^{-})\mathrm{Id}$ from Lemma 2.1 which, of course, remains valid over F. We see that $S_k(S_k)^{\mathrm{T}}$ is diagonizable if and only if $(S_{k-1})^{\mathrm{T}} S_{k-1}$ is. If E_λ, for λ in $\operatorname{spec}_F(\mathscr{L}_{k-1,k})$, denotes the eigenspace of $S_{k-1}(S_{k-1})^{\mathrm{T}}$ for λ, then the $(S_{k-1})^{\mathrm{T}} E_\lambda$ together with K, the kernel of $(S_{k-1})^{\mathrm{T}} S_{k-1}$, provide a decomposition into eigenspaces for $(S_{k-1})^{\mathrm{T}} S_{k-1}$, see Lemma 2.3. Hence $S_k(S_k)^{\mathrm{T}}$ is diagonizable over F. □

Of course $\operatorname{spec}_F(\mathscr{L}_{0,1})$ exists for trivial reasons so that one can use the proposition and proceed by induction. From the proof it is clear that the crucial point at each step is the structure of the isotropic subspace of the form induced by $S_k(S_k)^{\mathrm{T}}$.

3. Tactical decompositions

Here we are interested in the linear invariants of a tactical decomposition in a partially ordered set. We start by giving the relevant definitions. Let $\mathscr{S} = (\mathscr{P}, \mathscr{B}; \mathrm{I})$ be an incidence structure with 'point set' $\mathscr{P}$, 'block set' $\mathscr{B}$ and incidence relation $\mathrm{I} \leqslant \mathscr{P} \times \mathscr{B}$ in general.

Suppose that B is a partition of the block set into s classes $B_1, \ldots, B_j, \ldots, B_s$. Given some point $\not{p}$ in $\mathscr{P}$, let c_j be the number of blocks in B_j which are incident with $\not{p}$. The integer vector $c_B(\not{p}) = (c_1, \ldots, c_j, \ldots, c_s)$ is the *scheme* for $\not{p}$ relative to B.

Dually, if P is a partition of the point set into t classes $P_1, \ldots, P_i, \ldots, P_t$ then schemes for blocks are defined in the same way. A decomposition (P, B) of $\mathscr{S}$ is *tactical* if elements belonging to the same point or block class have the same scheme. In this case two matrices arise: the *point scheme* matrix C_B whose t rows are the schemes of point classes, and the *block scheme* matrix C_P whose s rows are the schemes of the block classes. In Chapter 1.1.3 of Dembowski's book these matrices are called simply the *incidence matrices* of the decomposition. The decomposition into singletons is trivially tactical and here $C_B\ (=C_P^{\mathrm{T}}) = S$ is the incidence matrix of $\mathscr{S}$.

For a given partition P of the point set denote by X the incidence matrix between points and point classes, that is $X_{\not{p},i} = 1$ if $\not{p}$ belongs to P_i and $X_{\not{p},i} = 0$ otherwise. The product $X^{\mathrm{T}}X =: N_P$ is the diagonal matrix of class sizes of P. For a partition of the block set the corresponding matrices are denoted by Y and N_B. A simple double counting argument – see also Section 1.1.3 in [5] – shows that $YC_P = S^{\mathrm{T}}X$ and $XC_B = SY$. Hence

LEMMA 3.1. (i) $YC_P = S^{\mathrm{T}}X$ *and* $XC_B = SY$;
(ii) $C_P^{\mathrm{T}}N_BC_P = X^{\mathrm{T}}SS^{\mathrm{T}}X$ *and* $C_B^{\mathrm{T}}N_PC_B = Y^{\mathrm{T}}S^{\mathrm{T}}SY$;
(iii) $N_BC_P = C_B^{\mathrm{T}}N_P$, $N_B(C_PC_B) = C_B^{\mathrm{T}}N_PC_B$. $N_P(C_BC_P) = C_P^{\mathrm{T}}N_BC_P$ *and* $N_P(C_BC_P)N_P^{-1} = (C_BC_P)^{\mathrm{T}}$.

The last equation shows that C_BC_P is symmetric in the inner product induced by N_P and so is diagonizable. Hence the collection of its eigenvalues, with multiplicities, will be called the *spectrum* of the decomposition and is denoted by spec(P, B). If (P, B) is the partition into singletons, then spec(P, B) is the spectrum for the whole structure, as defined earlier.

Various connections between spec(P, B) and spec$(\mathscr{S})$ can be established in general. Here we only need the following:

(a) Considering the decomposition (B, P) in the dual of $\mathscr{S}$, the non-zero parts of spec(P, B) and spec(P, B) are identical, by Lemma 2.3.
(b) If E_λ is an eigenspace of C_BC_P then Lemma 3.1 shows that XE_λ is an eigenspace of SS^{T} so that spec(P, B) is always part of spec$(\mathscr{S})$.

Let now $(\mathscr{L}, <)$ be a partially ordered set with rank function and suppose that $\mathscr{T}_k$ is a partition of $\mathscr{L}_k$ for each value of k. We denote the number of classes in $\mathscr{T}_k$ by t_k and set $\mathscr{T} = \{\mathscr{T}_k \mid k = 0, 1, 2, \ldots\}$. For a subset R of the rank values $\{|x| \mid x \in \mathscr{L}\}$ we say that $\mathscr{T}$ is an *R-tactical decomposition* of $\mathscr{L}$ if $(\mathscr{T}_k, \mathscr{T}_m)$ is tactical in $\mathscr{L}_{k,m}$ for every k and m in R.

Tactical decompositions arise naturally from groups of automorphisms: the orbits of the group on the elements of $\mathscr{L}$ is R-tactical for every R.

Taking a particular class of partially ordered sets, for instance $\mathscr{L} = 2^{\Omega}$, where Ω is a set of size n, one might ask whether *all* $\{1, \ldots, n\}$-tactical decompositions arise as orbits of permutation groups on Ω. I am not aware of examples which do not arise in this way. It may be of interest to answer this question. There are many examples if R is a 'small' subset of $\{1, \ldots, n\}$; these show also that many other questions can be phrased in terms of tactical decompositions:

Suppose that $\mathscr{B}$ is a collection of k-subsets from Ω such that $(\Omega, \mathscr{B})$ is a $t-(n, k, \mu)$ design on Ω. Let $\mathscr{T}_k$ be the partion of $\mathscr{L}_k$ into $\mathscr{B}$ and $\mathscr{L}_k \backslash \mathscr{B}$ and let $\mathscr{T}_k^*$ be the trivial partition (one class) of $\mathscr{L}_k^*$ for $k^* \leqslant t$ and arbitrary for the remaining values of k. Then $\mathscr{T}$ is $\{1, 2, \ldots, t, k\}$-tactical. Conversely, straightforward arguments show that tactical decompositions of this particular form define t-designs. One might take this observation as the general definition of a design on a partially ordered set with rank function.

For the remainder we consider only decompositions which are R-tactical for the whole set of rank values; for convenience we call such decompositions simply *tactical*. In what follows we determine their spectra. First we adapt the notation from above as follows: Considering $\mathscr{L}_k$ as the 'points' and $\mathscr{L}_{k+1}$ as the 'blocks' in $\mathscr{L}_{k,k+1}$ and supposing that a tactical decomposition $(\mathscr{T}_k, \mathscr{T}_{k+1})$ is given in $\mathscr{L}_{k,k+1}$, we put $C_k^+ := C_P$, $C_k^- := C_B$, $N_k := N_P$ and $N_{k+1} := N_B$. Furthermore, the incidence matrix between $\mathscr{T}_k$ and $\mathscr{L}_k$ is X_k so that $(X_k)^{\mathrm{T}} X_k = N_k$ and $\operatorname{spec}(\mathscr{T}_k, \mathscr{T}_{k+1})$ is formed by the eigenvalues of $(C_k^-)(C_k^+)$.

For a field F let the vector spaces corresponding to $\mathscr{L}_k$ and $\mathscr{T}_k$ be denoted by $F\mathscr{L}_k$ and $F\mathscr{T}_k$ respectively. Our notation is summarized best in Figure 1 in which all arrows of the same type commute, by Lemma 3.1.

$$
\begin{tikzcd}
\cdots F\mathscr{L}_{k-1} \arrow[r, shift left, dashed, "(S_{k-1})^{\mathrm{T}}"] \arrow[d, phantom, "\cup"] & F\mathscr{L}_k \arrow[l, shift left, "S_{k-1}"] \arrow[r, shift left, dashed, "S_k^{\mathrm{T}}"] & F\mathscr{L}_{k+1} \cdots \arrow[l, shift left, "S_k"] \\
\cdots F\mathscr{T}_{k-1} \arrow[u, shift left, "X_{k-1}"] \arrow[u, shift right, dashed, "X_{k-1}"'] \arrow[r, shift left, dashed, "C_{k-1}^+"] & F\mathscr{T}_k \arrow[u, shift left, "X_k"'] \arrow[u, shift right, dashed, "X_k"] \arrow[l, shift left, "C_{k-1}^-"] \arrow[r, shift left, dashed, "C_k^+"] & F\mathscr{T}_{k+1} \cdots \arrow[u, shift left, "X_{k+1}"'] \arrow[u, shift right, dashed, "X_{k+1}"] \arrow[l, shift left, "C_k^-"]
\end{tikzcd}
$$

Fig. 1.

LEMMA 3.2. *If $\mathscr{T}$ is a tactical decomposition in a join–meet regular partially ordered set $\mathscr{L}$ with parameters d_k^+, d_k^- and c_k, then for all k in $\{|x| \mid x \in \mathscr{L}\}$ we have*:

(i) $C_k^- C_k^+ - c_k(C_{k-1}^+)(C_{k-1}^-) = (d_k^+ - c_k d_k^-)\mathrm{Id}$, *and*

(ii) $(C_k^+)^{\mathrm{T}} N_{k+1} C_k^+ - c_k(C_{k-1}^-)^{\mathrm{T}} N_{k-1}(C_{k-1}^-) = (d_k^+ - c_k d_k^-) N_k$.

Proof. (i) Follow the arrows around Figure 1 and apply Lemma 2.1. (ii) Multiply the matrices in Lemma 2.1 by X_k on the right and by its transpose on the left. Now use Lemma 3.1(i). □

Now it becomes clear how the results of Section 2 generalize to spectra of tactical decompositions. Indeed, at the expense of clarity the results of Section 2 could have been stated in this general form realizing that spec($\mathscr{S}$) is the spectrum of the decomposition into singletons.

THEOREM 3.3. *If $\mathscr{T}$ is a tactical decomposition in a join–meet regular partially ordered set $\mathscr{L}$, let t_k denote the number of classes in $\mathscr{T}_k$. Then for all k in $\{|x| \,|\, x \in \mathscr{L}\}$* (i) *$t_{k-1} < t_k$ implies $d_k^+ \geqslant c_k d_k^-$, and* (ii) *$d_k^+ > c_k d_k^-$ implies $t_k \leqslant t_{k+1}$.*

Proof. Using Lemma 3.3 and taking into account that all spectra values are non-negative, the argument is identical to the one in Theorem 2.2. □

REMARKS. 1. In the case of 2^{Ω} the second part of 3.3 is Theorem 1 in Livingstone and Wagner [7]. For designs the inequality is the well-known result due to Block, Parker and others. For a general reference to orbit theorems, see [11].

(2) Note the curious converse in the first part of 3.3 and its combination with Theorem 2.2(ii). If there is *any* tactical decomposition in $\mathscr{L}$ with $t_{k-1} < t_k$ then $n_k \leqslant n_{k+1}$ unless $d_k^+ = c_k d_k^-$!

The next result is the description of the spectrum of $(\mathscr{T}_k, \mathscr{T}_{k+1})$ in terms of the number of partition classes.

THEOREM 3.4. *Let $\mathscr{T}$ be a tactical decomposition in a join–meet regular partially ordered set $(\mathscr{L}, <)$ and suppose that the numbers of partition classes $t_i = |\mathscr{T}_i|$ satisfy $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k$ up to some value of k. Then* spec$(\mathscr{T}_k, \mathscr{T}_{k+1})$ *consist of the values $\lambda_{0,k} > \lambda_{1,k} > \cdots > \lambda_{k,k}$ from* spec$(\mathscr{L}_{k,k+1})$ *where $\lambda_{i,k}$ has multiplicity $t_i - t_{i-1}$ for $0 \leqslant i \leqslant k$.*

Proof. The argument is identical to the one proving Theorem 2.4. □

We conclude with some remarks about the spectrum of tactical decompositions in $\mathscr{L}_{k,m}$ when $k < m$ are arbitrary. These are determined in a very similar fashion if $\mathscr{L}$ satisfies a chain condition: For all x in $\mathscr{L}_k$ and y in $\mathscr{L}_m$ the number of saturated chains $x = x_0 < x_1 < \cdots < x_{m-k} = y$ is either a constant (depending only on k and m) or is zero. This condition certainly holds in the subspace and subset lattices, but there are instances where this is not the case.

4. Symmetric Decompositions

Let $\mathscr{T}$, as before, be a tactical decomposition in the join–meet regular poset $\mathscr{L}$; we include the possibility that $\mathscr{L} = \mathscr{T}$ when $\mathscr{T}$ is the decomposition into singletons. The principal assumption of this section is that $t_k = t_{k+1}$ for some k so that there are equally many classes in $\mathscr{T}_k$ and in $\mathscr{T}_{k+1}$. Such a decomposition will be called *symmetric*.

This situation occurs automatically if $n_k = n_{k+1} = \mathrm{rank}_0(S_k)$, as for instance in the case of symmetric 2-designs with $k = 1$. However, there are many examples of symmetric decompositions when $n_k < n_{k+1}$. They have been studied in [1], [3] for subset lattices, in [8] for Steiner triple systems and in [2] for projective spaces.

From the previous section we may assume that the determinant Δ_k of $C_k^- C_k^+$ is known. Let $\pi_k = \det(N_k)$ and $\pi_{k+1} = \det(N_{k+1})$ be the products of the lengths of the classes in $\mathscr{T}_k$ and $\mathscr{T}_{k+1}$ respectively. Then, by Lemma 3.1(iii).

PROPOSITION 4.1. *If* $(\mathscr{T}_k, \mathscr{T}_{k+1})$ *is symmetric in* $\mathscr{L}$ *then* $\pi_k \Delta_k = \pi_{k+1}(\det(C_k^+))^2$ *and* $\pi_{k+1}\Delta_k = \pi_k(\det(C_k^-))^2$.

The determination of the class lengths in tactical decompositions seems to be a rather difficult problem. Some results for subset lattices can be found in [9] but these are specific to permutation groups. In this respect the proposition is a first step for symmetric decompositions.

Lemmas 3.1(iii) and 3.2(ii) yield much stronger results if one considers the basic equations as rational congruence relations for matrices.

Two rational square matrices A and B are *congruent* if there is a non-singular rational matrix C such that $C^{\mathrm{T}}AC = B$. Equivalently, A and B give rise to quadratic forms which differ only by a change of basis. This permits the application of the classical theory of quadratic forms due to Hasse and Minkowski for which we refer again to Section 1.1.3 in Dembowski's book, in particular with respect to the Hasse symbol H_p. A principal result is the fact that two matrices A and B are congruent if and only if $H_p\{A\} = H_p\{B\}$ for all primes p. A consequence of Lemma 3.2(ii) is therefore

THEOREM 4.2. *Let* $(\mathscr{T}_k, \mathscr{T}_{k+1})$ *be symmetric in* $\mathscr{L}$ *and suppose that* $\Delta_k \neq 0$. *Then* $H_p\{N_{k+1}\} = H_p\{(d_k^+ - c_k d_k^-)N_k + c_k(C_{k-1}^-)^{\mathrm{T}}N_{k-1}(C_{k-1}^-)\}$ *for all primes* p.

This statement is, of course, only useful if enough is known about $(C_{k-1}^-)^{\mathrm{T}}N_{k-1}(C_{k-1}^-)$. On occasions, however, this is automatic. Assume for instance $k = 1$. Then C_{k-1}^- is the row vector of class sizes of $\mathscr{T}_1$ and $N_0 = 1$. Hence $(C_{k-1}^-)^{\mathrm{T}}N_{k-1}(C_{k-1}^-) = N_1^{\mathrm{T}}JN_1$ where J is the all-one matrix and $(d_1^+ - c_1 d_1^-)N_1 + c_1(C_0^-)^{\mathrm{T}}N_0(C_0^-) = N_1^{\mathrm{T}}[(d_1^+ - c_1 d_1^-)N_1^{-1} + c_1 J]N_1$. The evaluation of the Hasse invariants for such a matrix becomes quite feasible, see Section 1.1.3 in Dembowski's book. We have therefore an extension of the Bruck–Ryser theorem on symmetric 2-designs.

COROLLARY 4.3. *If* $(\mathscr{T}_1, \mathscr{T}_2)$ *is symmetric in* $\mathscr{L}$ *with* $\Delta_1 \neq 0$ *then* $H_p\{N_2\} = H_p\{(d_1^+ - c_1 d_1^-)N_1^{-1} + c_1 J\}$ *for all primes p.*

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(Received, June 11, 1990)