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On Modular Homology in the Boolean Algebra

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Let Ω be a set, R a ring of characteristic $p > 0$, and denote by M_k the R-module with k-element subsets of Ω as basis. The *set inclusion map* $\partial: M_k \to M_{k-1}$ is the homomorphism which associates to a k-element subset Δ the sum $\partial(\Delta) = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_k$ of all its $(k-1)$-element subsets Γ_i. In this paper we study the chain

$$0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \dots M_k \leftarrow M_{k+1} \leftarrow M_{k+2} \dots \qquad (*)$$

arising from ∂. We introduce the notion of p-exactness for a sequence. If Ω is infinite we show that $(*)$ is p-exact for all prime characteristics $p > 0$. This result can be extended to various submodules and quotient modules, and we give general constructions arising from permutation groups with a finitary section. Two particular applications are the following: The orbit module sequence of such a permutation group on Ω is p-exact for every prime p, and we give a formula for the p-rank of the orbit inclusion matrix if the group has finitely many orbits on k-element subsets.

1. INTRODUCTION

Let Ω be a set and $2^{[\Omega]}$ the finite subsets of Ω. On $2^{[\Omega]}$ consider the map $\Delta \to \Gamma_1 + \Gamma_2 + \cdots + \Gamma_k$ which associates to a k-element subset Δ the formal sum of all its $(k-1)$-element subsets Γ_i. To make sense of this summation we regard subsets as elements of a module with coefficients in some ring. So if R is that ring, denote by $M_k = \{\sum r_\Delta \Delta \mid \Delta \subseteq \Omega,\ |\Delta| = k,$

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$r_\Delta \in R\}$ the module with k-element subsets of Ω as basis. The map is called the *set inclusion map* and denoted by ∂. In this paper we study the chain

$$0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \dots M_k \leftarrow M_{k+1} \leftarrow M_{k+2} \dots \tag{1}$$

arising from ∂, in particular when the characteristic of R is a prime $p \neq 0$. The first result (Theorem 3.4) shows that for infinite Ω all subsequences of the kind

$$\cdots \leftarrow M_{k-p} \leftarrow M_{k-p+i} \leftarrow M_k \leftarrow M_{k+i} \leftarrow M_{k+p} \cdots$$

are exact for arbitrary k and each i with $0 < i < p$. A sequence as in (1) satisfying this property is called *p-exact*.

In Section 4 we study sequences of submodules and quotient modules of the M_k. In particular, one of the main results (Theorem 4.4) shows that there is a general construction which yields p-exact chains of submodules derived from group rings and ideals of permutation groups with a finitary section on Ω. These are groups on Ω for which there is an infinite subset Ω^* so that each group element moves at most finitely points from Ω^*.

Two results can be derived from this. If G is a permutation group on Ω the orbit module M_k^G represents the G-orbits on k-element subsets of Ω. These are related to the augmentation ideal in RG, and in Theorem 4.5 we prove that the orbit module sequence of a finitary group is p-exact for every prime $p > 0$. The second application is Theorem 4.6. It gives a p-rank formula for the orbit inclusion matrix of a group with finitary section and finitely many orbits on k-element subsets.

Some of these results, as one might suspect, remain true when Ω is finite. A principal difference, however, is that the homology modules in the sequence (1) become non-trivial when k is about $|\Omega|/2$. As the techniques are rather different we felt it more appropriate to deal with the finite case separately. The paper [8] contains a very detailed analysis of the homological properties of (1) when Ω is finite.

Most results in the literature on the modular behaviour of inclusion maps consider only the case of finite Ω. Probably the most beautiful among these is the theorem of Wilson [13]. It applies to the situation when R is the ring of integers and says that for the inclusion map $M_k \to M_t$ (associating to Δ the sum of all its t-element subsets) there are suitable bases in M_k and M_t for which the map has diagonal matrix. It has the form $\mathrm{diag}(\binom{k}{t}, \binom{k-1}{t-1}, \dots, \binom{k-t+1}{1}, \binom{k-t}{0})$ where the ith entry has multiplicity $\dim(M_i) - \dim(M_{i-1})$. The proof of this result does not involve any kind of homological considerations even though the formula for the multiplicities suggests this connection.

From Wilson's theorem one can see that the p-rank of the inclusion map is some alternating sum of terms $\pm\dim(M_i)$. This coincides with the earlier modular rank formulae in [3–5, 14]. Comparing these to Theorem 4.5 it appears that the latter indeed is an infinite version of these results. Clearly, the alternating sums stem from the homological properties of the map and we believe that homology is the natural approach to modular inclusion maps.

2. INCLUSION MAPS

We begin by introducing our notation. Let R be an associative and commutative ring with 1 and Ω some infinite set. Then $R2^{[\Omega]}$ denotes the R-module of all formal sums $\Sigma r_\Delta \Delta$ in which Δ is a finite subset of Ω and $r_\Delta \in R$ is zero for all but finitely many Δ. For a natural number k the collection of all k-element subsets of Ω is denoted by $\Omega^{\{k\}}$ and $R\Omega^{\{k\}} \subset R2^{[\Omega]}$ denotes the submodule of expressions $\Sigma r_\Delta \Delta$ with $r_\Delta = 0$ unless $\Delta \in \Omega^{\{k\}}$. We abbreviate $R\Omega^{\{k\}}$ by M_k as the context usually is clear. Note in particular that $R2^{[\Omega]} = \oplus_{k \in \mathbb{N}} M_k$. Finally we assume that R acts faithfully on M_k so that the k-element subsets form a basis.

For $f = \Sigma r_\Delta \Delta \in R2^{[\Omega]}$ the *support* $\operatorname{supp}(f)$ is the union of all Δ for which $r_\Delta \neq 0$. Two elements f and g are said to be *disjoint* if $\operatorname{supp}(f)$ and $\operatorname{supp}(g)$ are disjoint sets.

The Boolean operations on $2^{[\Omega]}$ can be extended to products on $R2^{[\Omega]}$. The most important one for our purpose is the $\cup$-product: If $f = \Sigma f_\Delta \Delta$ and $g = \Sigma g_\Gamma \Gamma$, we define $f \cup g := \Sigma f_\Delta g_\Gamma (\Delta \cup \Gamma)$. It is not difficult to see that this definition turns $R2^{[\Omega]}$ into an associative ring with the empty set as identity. This algebra, usually in the context of finite Ω, has also been considered for instance in [7, 10, 11].

The *inclusion map* $\partial\colon R2^{[\Omega]} \to R2^{[\Omega]}$ is defined on a basis by $\partial(\Delta) := \Sigma_{\alpha \in \Delta}(\Delta \setminus \alpha)$ and extended to a homomorphism on $R2^{[\Omega]}$ by $\partial(\Sigma f_\Delta) := \Sigma f_\Delta \partial(\Delta)$. Clearly, this map restricts to homomorphisms $\partial\colon R\Omega^{\{k\}} \to R\Omega^{\{k-1\}}$. Very important is the product rule

$$\begin{aligned}&\text{if } f \text{ and } g \text{ are disjoint elements in } R2^{[\Omega]}\\ &\text{then } \partial(f \cup g) = \partial(f) \cup g + f \cup \partial(g)\end{aligned}$$

which can be verified easily. So ∂ behaves very much like ordinary differentiation. Therefore we often use the natural notation $f' := \partial(f)$ and $f^{(s)} = [f^{(s-1)}]'$.

For the remainder Ω denotes an infinite set unless explicitly stated otherwise.

INTEGRATION LEMMA 2.1. *If $m > k$ are integers suppose that $f \in M_k$ satisfies $f' = 0$. If $(m-k)!$ is invertible in R then there exists some $F \in M_m$ for which $F^{(m-k)} = f$.*

Proof. Select a set Δ of $m-k$ elements which is disjoint from $\operatorname{supp}(f)$. Put $F := [(m-k)!]^{-1}\Delta \cup \mathrm{f}$ and apply the product rule. ∎

LEMMA 2.2. *If $k < m$ and i are integers suppose that $0 < i < m-k$ and that $(m-k-i-1)!$ has an inverse in R. Then $\operatorname{Ker}(\partial^i) \cap M_{k+1} \subseteq \partial^{m-k-i}(M_m)$ implies that $\operatorname{Ker}(\partial^{i+1}) \cap M_{k+i+1} \subseteq \partial^{m-k-i-1}(M_m)$.*

Proof. As $\operatorname{Ker}(\partial^{i+1}) \cap M_{k+i+1} = \partial^{-1}(\operatorname{Ker}(\partial^i) \cap M_{k+i})$, let f be any element in $\operatorname{Ker}(\partial^i) \cap M_{k+i}$. By hypothesis, there is some $F \in M_m$ such that $F^{(m-k-i)} = f$. Put $h_1 = F^{(m-k-i-1)}$ and let h_2 be any element in $\partial^{-1}(f)$. So, $(h_1 - h_2)' = f - f = 0$, and using the integration lemma we find some $H \in M_m$ with $H^{(m-k-i-1)} = h_1 - h_2$. Since $[F-H]^{(m-k-i-1)} = h_1 - (h_1 - h_2)$, we have shown that $\partial^{-1}(f)$ belongs to $\partial^{m-k-i-1}(M_m)$. ∎

Taking both lemmas together, we obtain by induction the following fact:

Proposition 2.3. *Let Ω be an infinite set. If $0 \le k < m$ are integers for which $(m-k-1)!$ has an inverse in R, then $\operatorname{Ker}(\partial^i) \cap M_{k+1} \subseteq \partial^{m-k-i}(M_m)$ for all $0 \le i < m-k$.*

3. HOMOLOGY

From now on we suppose that R is a ring of characteristic $p \neq 0$. In particular, if p is a prime then R is an algebra over $GF(p)$.

The simple but crucial observation is that in this case $\partial^p\colon R2^{[\Omega]} \to R2^{[\Omega]}$ is the zero map: Let Δ be any set of size $d \ge p$. Then $\partial^p(\Delta) = c\Sigma\Gamma$ where summation runs over all $(d-p)$-element subsets Γ of Δ, and where c counts the number of chains $\Gamma = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_p = \Delta$. So c is $p! = 0$.

This leads us to investigate homology. First recall the usual definitions: if $\chi\colon A \to B$ and $\psi\colon B \to C$ are homomorphisms then the sequence $A \to B \to C$ is *homological* at B if $\operatorname{Ker}(\psi) \supseteq \chi(A)$, and *exact* if $\operatorname{Ker}(\psi) = \chi(A)$. A sequence $\cdots \leftarrow A_k \leftarrow A_{k+1} \leftarrow A_{k+2} \leftarrow A_{k+3} \leftarrow \cdots$ is *homological* (*exact*) if it has that property at every A_i.

Our objective is to study the sequence

$$0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \ldots M_k \leftarrow M_{k+1} \leftarrow M_{k+2} \ldots, \tag{1}$$

where as before M_j stands for $R\Omega^{\{j\}}$. Clearly, when R has characteristic $p = 2$, the sequence is homological. (Indeed, it is even exact, as we shall prove soon). However, for characteristic $p > 2$ we require a more general notion:

DEFINITION 3.1. If $2 \leq p$ is some integer, then the sequence $A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_{m-2} \leftarrow A_{m-1} \leftarrow A_m$ is *p-exact* (*p-homological*) if $A_k \leftarrow A_{k+i} \leftarrow A_{k+p}$ is exact (homological) for every $0 \leq k \leq m-p$ and every i, $1 \leq i < p$. (Arrows in the second sequence are the natural compositions of arrows in the first sequence.)

So 2-exactness is exactness in the usual meaning and $A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots \leftarrow A_k \leftarrow A_{k+1} \leftarrow A_{k+2} \ldots$ is 3-exact if and only if both $\cdots \leftarrow A_{k-3} \leftarrow A_{k-2} \leftarrow A_k \leftarrow A_{k+1} \leftarrow A_{k+3} \ldots$ and $\cdots \leftarrow A_{k-3} \leftarrow A_{k-1} \leftarrow A_k \leftarrow A_{k+2} \leftarrow A_{k+3} \ldots$ are exact. The connections to ordinary exactness can also be seen from the following proposition.

PROPOSITION 3.2. *Let* $\partial\colon A_i \to A_{i-1}$ *be homomorphisms and* $p \geq 3$ *an integer. If the sequence*

$$(*)\quad A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_{m-2} \leftarrow A_{m-1} \leftarrow A_m \text{ is } p\text{-exact then}$$

$$(**)\quad \partial(A_0) \leftarrow \partial(A_1) \leftarrow \partial(A_2) \ldots \partial(A_{m-2}) \leftarrow \partial(A_{m-1}) \leftarrow \partial(A_m)$$

is $(p-1)$-*exact*.

Proof. Evidently $(*)$ is p-homological if and only if $(**)$ is $(p-1)$-homological. Now suppose that $(*)$ is p-exact and consider $\partial(A_k) \leftarrow \partial(A_{k+i}) \leftarrow \partial(A_{k+p-1})$. Then $\partial^{p-i-1}(\partial(A_{k+p-1})) \subseteq \operatorname{Ker} \partial^i \cap \partial(A_{k+i}) = \{x \mid \partial^i(x) = 0 \text{ and } x = \partial(y) \text{ for some } y \in A_{k+i}\} = \partial(\partial^{p-i-1}(A_{k+p-1}))$. Hence $\partial^{p-i-1}(\partial(A_{k+p-1})) = \operatorname{Ker} \partial^i \cap \partial(A_{k+i})$ and so $(**)$ is $(p-1)$-exact. ∎

We now return to the sequence (1) from the beginning. To clarify the situation consider its first members. The module M_0 consists of all R-multiples of the empty set in Ω, and $0 \leftarrow M_0$ is the zero map.

LEMMA 3.3. *If the characteristic of R is a prime* $p \neq 0$ *then* $0 \leftarrow \cdots \leftarrow 0 \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_{p-1}$ *is p-exact.*

Proof. Clearly, $0 \leftarrow M_i \leftarrow M_j$ is homological for all $0 < i < j \leq p-1$. Exactness is equivalent to the surjectivity of $M_i \leftarrow M_j$ which follows from Corollary 2.5 in [9], see also [12]. ∎

However, p-exactness can be extended further:

THEOREM 3.4. *Let* Ω *be an infinite set and R a ring of prime characteristic* $p \neq 0$. *Then* $0 \leftarrow \cdots \leftarrow 0 \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_{m-1} \leftarrow M_m$ *is p-exact for any m.*

Proof. If R has characteristic p then $\partial^p = 0$ by the remark at the beginning of this section and so $\mathrm{Ker}(\partial^i) \cap M_{k+i} \supseteq \partial^{p-i}(M_{k+p})$ for all k and $i \le p$. Conversely, $\mathrm{Ker}(\partial^i) \cap M_{k+i} \supseteq \partial^{p-i}(M_{k+p})$ by Proposition 2.3. ∎

It is clear that this yields results on other submodules of $R2^{[\Omega]}$. An obvious consequence is the following fact.

COROLLARY 3.5. *The sequence* $R2^{[\Omega]} \leftarrow R2^{[\Omega]} \leftarrow \cdots \leftarrow R2^{[\Omega]}$ *is p-exact. Further, if* $B_k = \oplus_{0 \le i \le k} M_i$ *then* $0 \leftarrow \cdots \leftarrow 0 \leftarrow B_0 \leftarrow B_1 \leftarrow B_2 \cdots \leftarrow B_{m-1} \leftarrow B_m$ *is p-exact for any m.*

Finally note that Proposition 3.2 applies to the sequences in Theorem 3.4 and Corollary 3.5 and so these provide $(p-1)$-exact sequences in $R2^{[\Omega]}$.

4. AMPLE RINGS AND GROUP ACTIONS ON SUBSETS

Our aim now is to describe a general procedure which yields p-exact sequences in $R2^{[\Omega]}$ in a uniform way.

As before, R is a ring of prime characteristic $p \neq 0$. Permutations on Ω act naturally on $2^{[\Omega]}$ by $\Delta \to \Delta^g$ and this action is easily extended to $R2^{[\Omega]}$ by putting $g(\Sigma r_\Delta \Delta) := \Sigma r_\Delta \Delta^g$. The full symmetric group on Ω is denoted by S^Ω.

Let RS^Ω denote the group ring of S^Ω over R. This then also acts on $R2^{[\Omega]}$: for $a = \Sigma_{g \in G} r_g g$ we put $a\Delta := \Sigma_{g \in G} r_g \Delta^g$ and $af := \Sigma r_\Delta (a\Delta)$ for $f = \Sigma r_\Delta \Delta$ in $R2^{[\Omega]}$. Further, note that $\partial[a(\Sigma r_\Delta \Delta)] = a[\partial(\Sigma r_\Delta \Delta)]$ so that ∂ commutes with this action.

Let A be a subring of RS^Ω and denote $AM_k = \langle af \mid a \in A \text{ and } f \in M_k \rangle$ by A_k. Note that the inclusion map naturally restricts to $\partial: A_k \to A_{k-1}$. So we have a sequence of submodules

$$0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_k \leftarrow A_{k+1} \leftarrow A_{k+2} \ldots \tag{2}$$

and naturally the question arises: When is (2) p-exact?

DEFINITION 4.1. Let R be a ring of prime characteristic $p \neq 0$. Then the subring A of RS^Ω is said to be *ample* if $f \in A_k$ and $f' = 0$ for some k implies that $f = F^{(p-1)}$ for a suitable F in A_{k+p-1}.

In other words, A is ample precisely when the integration lemma holds in $0 \leftarrow \cdots \leftarrow 0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_{m-2} \leftarrow A_{m-1} \leftarrow A_m \ldots$.

PROPOSITION 4.2. *A is ample if and only if* $0 \leftarrow \cdots \leftarrow 0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_{m-2} \leftarrow A_{m-1} \leftarrow A_m$ is p-exact for all m.

Proof. One implication is obvious, and the sequence is p-homological in any case. So if A is ample we know that for any $f \in A_k$ with $f' = 0$, there is some F in A_{k+p-1} for which $f = F^{(p-1)}$. As the integration lemma holds in $0 \leftarrow \cdots \leftarrow 0 \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \ldots A_{m-2} \leftarrow A_{m-1} \leftarrow A_m$ we find that p-exactness follows verbatim as in Lemma 2.2 and Proposition 2.3. ▮

We give some examples of ample rings. If $a = \Sigma_{g \in G} a_g g$ is an element of RS^{Ω} call $\mathrm{Fix}(a) = \{\omega \mid \omega \in \Omega,\ a_g \neq 0$ implies that $\omega^g = \omega\}$ the *fixed point set* of a.

PROPOSITION 4.3. *A is ample provided that $A_k = \langle af \mid a \in A,\ f \in M_k$ for which $|Fix(a) \setminus supp(af)| \geq p - 1\rangle$ for all k.*

Proof. Let h be in A_k with $h' = 0$ and suppose that $h = af$ where $|\mathrm{Fix}(a) \setminus \mathrm{supp}(h)| \geq p - 1$. So we are able to select a $(p-1)$-set Γ in $\mathrm{Fix}(a)$ disjoint from h. Then $[h \cup \Gamma]^{(p-1)} = (p-1) \mid h$ by the product rule. It remains to show that $h \cup \Gamma$ belongs to A_{k+p-1}. So if $a = \Sigma a_g g$ and $f = \Sigma f_{\Delta} \Delta$ then, as $\Gamma = \Gamma^g$ if $a_g \neq 0$, we have $(af) \cup \Gamma = (\Sigma_{g,\Delta} a_g f_{\Delta} \Delta^g) \cup \Gamma^g = \Sigma_g a_g (\Sigma_{\Delta} f_{\Delta} (\Delta \cup \Gamma))^g = a(\Sigma_{\Delta} f_{\Delta} (\Delta \cup \Gamma)) = a(f \cup \Gamma)$ in A_{k+p-1}. So A is ample. ▮

While these two observations are far from a complete description of ampleness—for instance, which subrings of ample rings are ample ?—they are nevertheless general enough to investigate groups with a *finitary section*. By this we simply mean a permutation group G on Ω with an infinite subset Ω^*, such that each element of G moves at most finitely many points from Ω^*.

THEOREM 4.4. *Let $G \subseteq S^{\Omega}$ have a finitary section on Ω and let R be a ring of prime characteristic $p \neq 0$. If A is a left ideal in the group ring RG, then $0 \leftarrow \cdots 0 \leftarrow A_0 \leftarrow A_1 \cdots \leftarrow A_{m-1} \leftarrow A_m$ and $0 \leftarrow \cdots 0 \leftarrow M_0/A_0 \leftarrow M_1/A_1 \cdots \leftarrow M_{m-1}/A_{m-1} \leftarrow M_m/A_r$ are p-exact and G-invariant sequences for every m.*

Proof. A being a left ideal means that the two sequences are G-invariant. As G has a finitary section the condition of Proposition 4.3 holds in A. So $0 \leftarrow \cdots 0 \leftarrow A_0 \leftarrow A_1 \cdots \leftarrow A_{m-1} \leftarrow A_m$ is p-exact by Proposition 4.2. As in the case of ordinary exact sequences, the p-exactness of $0 \leftarrow \cdots 0 \leftarrow A_0 \leftarrow A_1 \cdots \leftarrow A_{m-1} \leftarrow A_m$ and of $0 \leftarrow \cdots 0 \leftarrow M_0 \leftarrow M_1 \ldots M_{m-1} \leftarrow M_m$—see Theorem 3.4—implies that the quotient sequence is also p-exact. ▮

An important kind of ideal in the group algebra is the augmentation ideal $\mathrm{Aug}(G) = \{a \mid a = \Sigma a_g g$ in RG, $\Sigma a_g = 0\}$. When we apply Theorem 4.4 to this ideal one observes without difficulty that the corresponding submodules $A_k = \mathrm{Aug}(G) M_k$ satisfy the following: Two cosets $\Gamma + A_k$ and $\Gamma^* + A_k$ for k-element sets Γ and Γ^* coincide if and only if Γ and Γ^* belong to the same G-orbit. Furthermore, the distinct cosets of this form are a basis of the quotient module—see also the lemma in [2]. Therefore we call $M_k^G := M_k/A_k$ the *orbit module* of G on $\Omega^{\{k\}}$.

THEOREM 4.5. *Let $G \subseteq S^\Omega$ have a finitary section on Ω and let R be a ring of prime characteristic $p \neq 0$. Then $0 \leftarrow \cdots \leftarrow 0 \leftarrow M_0^G \leftarrow M_1^G \cdots \leftarrow M_{m-1}^G \leftarrow M_m^G$ is p-exact for all m.*

As an application of this result consider the case when the group happens to have finitely many orbits on $\Omega^{\{k\}}$. Denote their number by $n_k(G)$. It is well known that $n_t(G) \leq n_k(G)$ when $t \leq k$. Define the *orbit inclusion matrix* $W_{tk}(G, \Omega)$ as the matrix whose columns are indexed by G-orbits on $\Omega^{\{k\}}$, its rows by G-orbits on $\Omega^{\{t\}}$, and with (i, j)-entry, for a fixed k-set Γ in the j^{th} orbit, counting the number of t-subsets $\Delta \subseteq \Gamma$ belonging to the i^{th} orbit.

Clearly, we can view $W_{tk}(G, \Omega)$ as a matrix over any field and in particular over $GF(p)$. As a corollary to the last theorem we obtain a formula for its rank.

THEOREM 4.6. *Let $G \subseteq S^\Omega$ have a finitary section on Ω and finitely many orbits on $\Omega^{\{k\}}$. If p is a prime and $k - p < t < k$, then the rank of $W_{tk}(G, \Omega)$ over $GF(p)$ is $n_t(G) - n_{k-p}(G) + n_{t-p}(G) - n_{k-2p}(G) + n_{t-2p}(G) - \cdots$.*

Proof. Let R be $GF(p)$. From the description of M_k^G above it follows that $\partial^{(k-t)}: M_k^G \to M_t^G$ has matrix $(k-t)! W_{tk}(G, \Omega)$ when entries are interpreted over R. Therefore consider the sequence $0 \leftarrow \cdots \leftarrow M_{k-2p}^G \leftarrow M_{t-p}^G \leftarrow M_{k-p}^G \leftarrow M_t^G \leftarrow M_k^G$. The dimension of M_i^G is $n_i(G)$ and so $\operatorname{rank}_p W_{tk}(G, \Omega) = \dim(\partial^{(k-t)} M_k^G) = \dim(\operatorname{Ker} \partial^{(t+p-k)} \cap M_t^G) = n_t(G) - \dim(\partial^{(t-k+p)} M_t^G)$ as the sequence is exact. The remainder follows by induction. ∎

Remarks. (1) This result can be viewed as a generalisation of the rank formulae [4, 5, 13, 14] for finite Ω and $G = 1$. In [8] we obtain similar results for finite Ω and groups whose order is not divisible by p. For finite Ω and large p the equivalent of Theorem 4.6 then includes the first theorem in [6]. It should also be possible to avoid the restriction $k - p < t < k$ by considering more general sequences in a similar way.

(2) The submodule structure of M_k has recently [1] been worked out for the symmetric groups and various characteristics. We thank Alan Camina and David Evans for useful comments.

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