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CENTRALIZER ALGEBRAS FOR AN
AUTOMORPHISM GROUP OF A
FINITE INCIDENCE STRUCTURE

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SUNTO. — Questa nota è basata su una Conferenza tenuta presso l'Università di Milano nel gennaio 1986. Si vuol mostrare che l'Algebra centralizzante sui blocchi per l'azione di un gruppo di automorfismi di una struttura di incidenza determina completamente l'Algebra centralizzante per l'azione sui punti della struttura.

1. - INTRODUCTION.

When the elements in a permutation group \mathcal{G} on a (finite) set \mathcal{P} are represented as permutation matrices, the centralizer algebra for this action is the set of all matrices commuting with every element in \mathcal{G} . The orbits of \mathcal{G} on the pairs of points in \mathcal{P} determine a basis of this algebra and, conversely, any explicit representation of it gives complete information about these orbits. For this reason the centralizer algebra associated to a permutation group contains important information about the action of the group. The dimension of this algebra, for instance, is the permutation rank of \mathcal{G} and from a basis of it all systems — if any — of imprimitivity can be retrieved.

In this note we are concerned with an automorphism group \mathcal{G} of an incidence structure \mathcal{S} on the point set \mathcal{P} . Thus \mathcal{G} also has a representation on the blocks of \mathcal{S} and associated to it is the centralizer algebra for this action. Under the assumption that an incidence matrix for \mathcal{S} has linear rank equal to $|\mathcal{P}|$ (in characteristic zero) we show that the centralizer algebra for the block action determines the centralizer algebra for the point action. Thus, orbits on pairs of

blocks determine orbits on pairs of points and we show in particular that the permutation rank on points is bounded from above by the permutation rank of the point action.

2. - NOTATION.

Let \mathcal{G} be a permutation group on some finite set \mathcal{P} of $v := |\mathcal{P}|$ points. Once the points in \mathcal{P} are arranged in some way the elements in \mathcal{G} are represented as $v \times v$ permutation matrices. Let now $C_P(\mathcal{G})$ be the set of all $v \times v$ matrices over \mathbb{Q} which commute with every permutation matrix in \mathcal{G} . These matrices form an algebra, called the *centralizer algebra* of the permutation group $(\mathcal{G}, \mathcal{P})$. Its dimension is the number of $r_P(\mathcal{G})$ of \mathcal{G} -orbits on pairs of points.

Let now \mathcal{B} , with $b := |\mathcal{B}|$, be a finite set disjoint from \mathcal{P} . Its elements are called *blocks*. An *incidence relation* is a non-empty subset \mathcal{I} of $\mathcal{P} \times \mathcal{B}$ and the triple $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$ is called an *incidence structure*. Once also the blocks are arranged in some fashion then \mathcal{S} can be represented by the incidence matrix S whose rows are indexed by points and whose columns are indexed by blocks. For $(p, b) \in \mathcal{P} \times \mathcal{B}$ the $(S_{p,b})$ -entry is 1 if $(p, b) \in \mathcal{I}$ and is 0 otherwise.

An automorphism of \mathcal{S} is a pair (G, H) of permutation matrices, G of size $v \times v$ and H of size $b \times b$, such that $GS = HS$. Equivalently, (G, H) represents a pair of permutations preserving the incidence relation in \mathcal{S} . The group of all automorphisms of \mathcal{S} is denoted by $\text{Aut}(\mathcal{S})$. When \mathcal{G} is a subgroup of $\text{Aut}(\mathcal{S})$ let as above $C_P(\mathcal{G})$ denote the centralizer algebra of \mathcal{G} on points and let $C_B(\mathcal{G})$ be the centralizer algebra of \mathcal{G} in its action on blocks. Here $\dim(C_B(\mathcal{G})) = r_B(\mathcal{G})$ is the number of \mathcal{G} -orbits on pairs of blocks. As a general reference to centralizer algebras see § 28, 29 in [3] or chapter 1 in [1].

3. - CENTRALIZER ALGEBRAS.

Let now $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$ be a finite incidence structure. We shall assume that the incidence matrix S has linear rank equal to v over the field \mathbb{Q} . This assumption holds for a great number of incidence structures, including 2-designs, connected graphs (with the exception

of bipartite graphs), all subsets of a finite set with containment as incidence and subspace inclusion in a finite affine or projective space. See also § 3 in [2].

THEOREM. - Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}; \mathcal{I})$ be a finite incidence structure with incidence matrix S of rank equal to $|\mathcal{P}|$ over \mathbb{Q} . Let $\bar{S} = S^T \cdot (S \cdot S^T)^{-1}$. For a subgroup $\mathcal{G} \subseteq \text{Aut}(\mathcal{S})$ let $C_P(\mathcal{G})$ and $C_B(\mathcal{G})$ be the centralizer algebras of \mathcal{G} on points and on blocks respectively. Define $\Phi: C_P(\mathcal{G}) \rightarrow C_B(\mathcal{G})$ by $\Phi(M) = \bar{S}MS$ and $\bar{\Phi}: C_B(\mathcal{G}) \rightarrow C_P(\mathcal{G})$ by $\bar{\Phi}(N) = SNS$. Then Φ is an injective algebra homomorphism. The map $\bar{\Phi}$ is a linear surjection and, if $\bar{S} \cdot S$ belongs to the centre of $C_B(\mathcal{G})$, also an algebra homomorphism.

PROOF. - Note first that $S \cdot S^T$ is indeed non-singular. For $y \cdot S \cdot S^T = 0$ implies $(yS) \cdot (yS)^T = 0$ so that $y \cdot S = 0$ and, since $\text{rank}(S) = v$, also $y = 0$. Thus $S \cdot \bar{S} = 1_v$ and $\bar{S} \cdot S = 1_b$ if and only if $v = b$. When (G, H) is an automorphism of \mathcal{S} then $GS = SH$, hence $S^T G^T = H^T S^T$. This implies $\bar{S}G = S^T(S \cdot S^T)^{-1} \cdot G = S^T(G^T \cdot S S^T)^{-1} = S^T(SS^T G^T)^{-1} = S^T G(SS^T)^{-1} = H \bar{S}$. That Φ and $\bar{\Phi}$ are linear maps is immediately clear. If M is in $C_P(\mathcal{G})$ then $H\Phi(M)H^{-1} = H\bar{S}MSH^{-1} = \bar{S}GMG^{-1}S = \Phi(M)$ belongs to $C_B(\mathcal{G})$ and for N in $C_B(\mathcal{G})$ we have $G\bar{\Phi}(N)G^{-1} = GSN\bar{S}G^{-1}S = SGNG^{-1}\bar{S} = \bar{\Phi}(N)$ in $C_P(\mathcal{G})$. Since $\Phi(MM') = \bar{S}MM'S = \bar{S}MS\bar{S}M'S = \Phi(M)\Phi(M')$ we have that Φ is an algebra homomorphism. Similarly, if $\bar{S}S$ commutes with every N' in $C_B(\mathcal{G})$, then $\bar{\Phi}(NN') = SNN'\bar{S} = SNN'\bar{S}S\bar{S} = SN\bar{S}SN'\bar{S} = \bar{\Phi}(N)\bar{\Phi}(N')$ so that $\bar{\Phi}$ is an algebra homomorphism. \square

Let $D_P(\mathcal{G})$ be the subalgebra of diagonal matrices in $C_P(\mathcal{G})$. The orbits of \mathcal{G} on points — say $n = n_P(\mathcal{G})$ in number — correspond to \mathcal{G} -orbits on pairs of equal points and thus give rise to diagonal matrices $\Delta_{P,1}, \dots, \Delta_{P,n}$ in $D_P(\mathcal{G})$. These matrices form a basis of $D_P(\mathcal{G})$ so that $\dim D_P(\mathcal{G}) = n_P(\mathcal{G})$. It is a simple but important observation that the $\Delta_{P,i}$'s can be constructed whenever any spanning set for $D_P(\mathcal{G})$ is given. Thus an explicit representation of $D_P(\mathcal{G})$ determines the orbits of \mathcal{G} on \mathcal{P} and vice versa. Using the same notation, let $D_B(\mathcal{G})$ denote the diagonal subalgebra in $C_B(\mathcal{G})$

of dimension $n_B(\mathcal{G})$, the number of \mathcal{G} -orbits on \mathcal{B} . Notice now that both ϕ and $\bar{\phi}$ (as in the theorem) restrict to these diagonal subalgebras. As $\bar{\phi}\phi$ is the identity of $C_P(\mathcal{G})$ the map ϕ in particular is surjective. This gives

Corollary 1: (theorem 3.1 in [2]).

Under the general assumption of the theorem, the orbits of \mathcal{G} on \mathcal{B} determine the orbits of \mathcal{G} on \mathcal{P} and $n_P(\mathcal{G}) \leq n_B(\mathcal{G})$.

Consider now the orbits of \mathcal{G} on pairs of distinct points. They give rise to *orbital matrices* $\Gamma_{P,1}, \dots, \Gamma_{P,m}$ in $C_P(\mathcal{G})$ who, together with the orbit matrices $\Delta_{P,1}, \dots, \Delta_{P,m}$, form a basis of $C_P(\mathcal{G})$. Thus $m + n = \dim C_P(\mathcal{G}) = r_P(\mathcal{G})$, the rank of \mathcal{G} on points. Note that the diagonal matrices in $C_P(\mathcal{G})$ can be constructed from the $\Gamma_{P,i}$'s. Again we observe that an explicit representation of $C_P(\mathcal{G})$ determines the orbitals of \mathcal{G} on points and vice versa. In the same way $C_B(\mathcal{G})$ is spanned by orbital matrices $\Gamma_{B,1}, \dots$ for \mathcal{G} in its action on pairs of distinct blocks, together with the diagonal matrices in $C_B(\mathcal{G})$. Hence we have

Corollary 2:

Under the general assumption of the theorem, the orbits of \mathcal{G} on pairs of blocks determine the orbits of \mathcal{G} on pairs of points and $r_P(\mathcal{G}) \leq r_B(\mathcal{G})$.

SUMMARY. — This note is based upon a talk I gave at the University of Milan in January, 1986. We shall prove that the centralizer algebra on blocks for the action of an automorphism group of an incidence structure determines completely the centralizer algebra for the action on the points of the structure.

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