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Decompositions of Modules Associated to Finite Partially Ordered Sets

JOHANNES SIEMONS

For a finite partially ordered set $\mathbb{L}$ and a field F, let $F\mathbb{L}$ be the associated vector space with $\mathbb{L}$ as basis. In the case of ranked posets this space decomposes into eigenspaces under the maps afforded by the order relation. In this note we show how to construct generating sets for such decompositions and discuss their combinatorial significance.

1. ASSOCIATED MODULES

We consider finite partially ordered sets $(\mathbb{L}, <)$ with unique minimal element 0 and with rank function $|\,|: \mathbb{L} \to \mathbb{N}$; that is, $|x| = r$ is the length of any saturated chain $0 < x_1 < \cdots < x_r = x$. Denote the set of elements of rank k by $\mathbb{L}_k$.

For a field F the vector space $F\mathbb{L}_k$ consists of all formal sums $\sum f_x x$ with f_x in F and x in $\mathbb{L}_k$. Also put $F\mathbb{L} := \oplus_k F\mathbb{L}_k$. The order relation on $\mathbb{L}$ gives rise to the maps $^+: F\mathbb{L}_k \to F\mathbb{L}_{k+1}$ and $^-: F\mathbb{L}_k \to F\mathbb{L}_{k-1}$ which are defined on a basis of $F\mathbb{L}_k$ by $x^+ = \sum_{x<y} y$ (with y in $\mathbb{L}_{k+1}$) and $x^- = \sum_{x>z} z$ (with z in $\mathbb{L}_{k-1}$). These maps thus lead to the chain

$$F\mathbb{L}_0 \underset{-}{\overset{+}{\rightleftarrows}} F\mathbb{L}_1 \underset{-}{\overset{+}{\rightleftarrows}} F\mathbb{L}_2 \underset{-}{\overset{+}{\rightleftarrows}} \cdots \underset{-}{\overset{+}{\rightleftarrows}} F\mathbb{L}_k \underset{-}{\overset{+}{\rightleftarrows}} F\mathbb{L}_{k+1} \underset{-}{\overset{+}{\rightleftarrows}} F\mathbb{L}_{k+2} \cdots. \tag{1}$$

Now observe that $^+$ and $^-$ are adjoint to each other in the standard inner products. Hence the maps $^{+-}$ and $^{-+}$ are self-adjoint or symmetric. Correspondingly—at least for suitable fields—each $F\mathbb{L}_k$ decomposes into eigenspaces under $^{+-}$ and $^{-+}$. These decompositions are the subject of this note. They appear to be important objects, as they are defined entirely in terms of the order relation in $\mathbb{L}$.

Below we construct generating sets when $\mathbb{L}$ is the Boolean lattice of subsets of a finite set. The construction seems to rely mostly on the fact that $\mathbb{L}$ has a large group of automorphisms. For this reason I expect that finite projective spaces can be treated analogously.

2. EIGENSPACE DECOMPOSITIONS

Regarding the sequence (1) above, we fix some value of k and suppose that $\oplus_\lambda E_\lambda = F\mathbb{L}_k \leftrightarrow F\mathbb{L}_{k+1} = \oplus_{\lambda^*} E^*_{\lambda^*}$ are decompositiosn into eigenspaces E_λ of $^{+-}$ and $E^*_{\lambda^*}$ of $^{-+}$ respectively. The lemma below shows that we can arrange the sums in such a way that $\lambda = \lambda^*$ with isomorphisms $E^*_{\lambda^*} = (E_\lambda)^+$ and $E_\lambda = (E^*_{\lambda^*})^-$ unless $\lambda = 0 = \lambda^*$.

LEMMA 1. *Let $\alpha: V \to V^*$ and $\beta: V^* \to V$ be linear maps between vector spaces. Denote by E_λ the eigenspace of the map $\beta\alpha: V \to V$ and by E^*_λ the eigenspace of the map $\alpha\beta: V^* \to V^*$ for the same value λ in both cases. If $\lambda \neq 0$ then $\alpha: E_\lambda \to E^*_\lambda$ and $\beta: E^*_\lambda \to E_\lambda$ are isomorphisms.*

(The proof is very easy: for x in E_λ we have $\alpha\beta(\alpha x) = \alpha(\beta\alpha x) = \lambda\alpha x$ so that $\alpha: E_\lambda \to E^*_\lambda$, even injectively as long as $\lambda \neq 0$. Now apply the same argument to β.)

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Comparing the decomposition $\oplus_\lambda E_\lambda = F\mathbb{L}_k \leftrightarrow F\mathbb{L}_{k+1} = \oplus_{\lambda^*} E^*_{\lambda^*}$ to those of adjacent spaces $F\mathbb{L}_{k-1} \leftrightarrow F\mathbb{L}_k$ or $F\mathbb{L}_{k+1} \leftrightarrow F\mathbb{L}_{k+2}$ we note that in general there may be no relation at all. However, frequently a linear *recurrence relation* of the type

$$x^{+-} - a_k x = b_k x^{-+} \qquad (a_k, b_k \neq 0 \text{ in } F, \text{ all } x \text{ in } \mathbb{L}_k \text{ and all } k) \tag{2}$$

holds. For instance, this is the case in the Boolean lattice, in projective spaces or in posets associated to 2-designs. Partially ordered sets satisfying (2) were considered in [6] and have been the motivation for this note.

In this situation it is immediate that $F\mathbb{L}_{k-1} \leftrightarrow F\mathbb{L}_k$ and $F\mathbb{L}_k \leftrightarrow F\mathbb{L}_{k+1}$ give rise to the *same* eigenspace decomposition for each $F\mathbb{L}_k$. We therefore write $F\mathbb{L}_k = \oplus_\lambda E_{\lambda,k}$ unambiguously. Also let $x \to x^{+r}$ denote the r-fold application of the map $x \to x^+$.

PROPOSITION 1. *Suppose that $\mathbb{L}$ satisfies* (2) *above. Then the eigenvalues of the map* $^{+-}: F\mathbb{L}_k \to F\mathbb{L}_k$ *are at most $k+1$ integers $\lambda_{i,k}$ for $0 \leq i \leq k$ and every k. Furthermore, if $|\mathbb{L}_0| \leq |\mathbb{L}_1| \leq \cdots \leq |\mathbb{L}_k| \leq |\mathbb{L}_{k+1}|$ then the $\lambda_{i,k}$ can be arranged as $\lambda_{0,k} \geq \lambda_{1,k} \geq \lambda_{2,k} \geq \cdots \geq \lambda_{i,k} \geq \cdots \geq \lambda_{k-1,k} \geq \lambda_{k,k}$, where $\lambda_{i,k}$ has multiplicity $|\mathbb{L}_i| - |\mathbb{L}_{i-1}|$, independently of k. The corresponding eigenspace satisfies $E_{\lambda_{i,k}} = (E_{\lambda_{i,i}})^{+(k-i)}$.*

For details, see [6, Theorem 2.4].

3. Generating Sets in the Boolean Lattice

We give an explicit description of generators for the eigenspaces $E_{\lambda_{i,k}}$ above when $\mathbb{L}$ is the power set of some finite set Ω of size n. We only consider the case when $2k \leq n$. By Proposition 1 above, it will be sufficient to determine $E_{\lambda_{t,t}}$ for every t. Let F be the field of rational numbers and consider the map $^{+-}: F\mathbb{L}_t \leftrightarrow F\mathbb{L}_t$. As $E_{\lambda_{t,t}}$ is an eigenspace of $^{-+}: F\mathbb{L}_t \leftrightarrow F\mathbb{L}_t$ and as $E_{\lambda_{0,t}} + E_{\lambda_{1,t}} + \cdots + E_{\lambda_{(t-1),t}}$ is isomorphic to $F\mathbb{L}_{t-1}$ it follows that, alternatively, $E_{\lambda_{t,t}}$ is the kernel of $^-: F\mathbb{L}_t \to F\mathbb{L}_{t-1}$. So, for instance, $E_{\lambda_{0,0}}$ is $F\mathbb{L}_0 \simeq F$, $E_{\lambda_{1,1}}$ is $\langle \alpha - \beta \mid \alpha, \beta \in \Omega \rangle$ and $E_{\lambda_{2,2}}$ is generated by elements of the form $e_1 - e_2 + e_3 - e_4$, where e_1, e_2, e_3, e_4 when viewed as a graph form a 4-cycle.

We extend the Boolean operations on Ω to turn $F\mathbb{L}$ into an associative algebra (see [5] and Mnukhin [4], also in these proceedings): for $f = \sum_{x \in \mathbb{L}} f_x x$ in $F\mathbb{L}$ and y in $\mathbb{L}$ we put $f \cup y := \sum_{x \in \mathbb{L}} f_x\,(x \cup y)$. The $\cup$-product can be extended linearly to the whole of $F\mathbb{L}$, which is now an associative algebra with identity. We say that $g = \sum_{y \in \mathbb{L}} g_y y$ is *disjoint* from f if $f_x \neq 0 \neq g_y$ implies that x and y are disjoint sets. An important fact is that disjoint elements satisfy the *product rule* $(f \cup g)^- = f \cup g^- + f^- \cup g$.

PROPOSITION 2. *Let M be a generating set for $E_{\lambda_{t-1,t-1}}$. For points $\alpha \neq \beta$ in Ω denote the set of all elements in M disjoint from $\{\alpha, \beta\}$ by $M_{\alpha\beta}$. Then $M^* = \{(\alpha - \beta) \cup m \mid \alpha \neq \beta \in \Omega$ and $m \in M_{\alpha\beta}\}$ is a generating set for $E_{\lambda_{t,t}}$.*

PROOF. Let U be the subspace generated by M^*. The expressions in M^* involve t-sets only, and $((\alpha - \beta) \cup m)^- = (\alpha - \beta) \cup m^- + (\alpha - \beta)^- \cup m = 0$; hence U is contained in $E_{\lambda_{t,t}}$. The symmetric group acts on $F\mathbb{L}$ by $f = \sum_{x \in \mathbb{L}} f_x x \to f^g = \sum_{x \in \mathbb{L}} f_x x^g$ for g in $\mathrm{Sym}(\Omega)$. We show that U is invariant under $\mathrm{Sym}(\Omega)$. Let g be a permutation: then $((\alpha - \beta) \cup m)^g = ((\alpha^g - \beta^g) \cup m^g)$. As m^g belongs to $E_{\lambda_{t-1,t-1}}$ it can be written as a combination of elements from M which, furthermore, are all disjoint from $\{\alpha^g, \beta^g\}$. Hence $((\alpha - \beta) \cup m)^g$ is in U.

Now let x be in $E_{\lambda_{t,t}}$ and let $\alpha \neq \beta$ be any two points of Ω. We write x uniquely as $x = \alpha \cup a + \beta \cup b + \{\alpha, \beta\} \cup c + d$, where $a, b \in F\mathbb{L}_{t-1}$, $c \in F\mathbb{L}_{t-2}$ and $d \in F\mathbb{L}_t$ are all disjoint from $\{\alpha, \beta\}$. If g is the transposition interchanging α and β but fixing the remaining points, then $x^g = \beta \cup a + \alpha \cup b + \{\alpha, \beta\} \cup c + d$, so that $x - x^g = (\alpha - \beta) \cup (a - b)$. Since $0 = (x - x^g)^- = (0) \cup (a - b) + (\alpha - \beta) \cup (a - b)^-$ it follows that $(a - b)^- = 0$ and so $x - x^g$ belongs to U. We write this as $x \equiv x^g \bmod(U)$. As g is an arbitrary transposition, it follows that $x \equiv x^g \bmod(U)$ holds for any g in $\mathrm{Sym}(\Omega)$. As $\mathrm{Sym}(\Omega)$ is finite the coset $x + U$ must therefore contain a vector fixed by $\mathrm{Sym}(\Omega)$. However, the only such vectors are in $E_{\lambda_{0,t}}$ (the 1-dimensional space spanned by the sum over all t-subsets). Hence 0 belongs to $x + U = U$, and the proof is complete. □

The generating sets given above for $E_{\lambda_{0,0}}$, $E_{\lambda_{1,1}}$ and $E_{\lambda_{2,2}}$ evolve in this pattern from the vector in $E_{\lambda_{0,0}}$. Continuing in the same way we construct what one might call the *standard generating set* $S_{t,t}$ for each $E_{\lambda_{t,t}}$. Thus take x in $S_{t,t}$. Then x involves 2^t sets with coefficients either 1 or -1 and the union of those sets appearing in x with non-zero coefficient forms as set Δ, of size $2t$. We put $\sigma_{x,r} := \sum_{\Gamma \subset \Omega \setminus \Delta, |\Gamma| = r} \Gamma$ and define $S_{t,k} = \{x \cup \sigma_{x,r} \mid x \text{ in } S_{t,t} \text{ and } r = k - t\}$.

PROPOSITION 3. *For $t \leq k \leq n/2$ the space $E_{\lambda_{t,k}}$ is generated by $S_{t,k}$.*

PROOF. We only need to consider the case $t < k$. By Proposition 1 above $E_{\lambda_{t,k}}$ is generated by $\{x^{+r} \mid x \text{ in } S_{t,t}\}$, where $r = k - t$. Hence the result follows from the formula

$$x^{+r} = r! \cdot (x \cup \sigma_r), \qquad \text{where } \sigma_r = \sigma_{x,r}, \tag{3}$$

which we now prove. By induction we assume $x^{+(r-1)} = (r-1)! \cdot (x \cup \sigma_{r-1})$ and compute $(x \cup \sigma_{r-1})^+$. The latter is equal to $r \cdot (x \cup \sigma_r) + (x^{(+)} \cup \sigma_{r-1})$ where ${}^{(+)}$ denotes the map ${}^+$ when applied to subsets a of Δ; that is, $a^{(+)} = \sum_{\alpha \in \Delta \setminus a} a \cup \alpha$. But it is easy to see that ${}^{(+)}$ and ${}^-$, on the module of t-subsets of a set of size $2t$, have the same kernel. Thus, as $x^- = 0$, also $x^{(+)} = 0$ and so (3) holds. □

4. WEIGHTS

The *weight* of a vector x in $F\mathbb{L}$ is the number $w(x)$ of sets appearing in x with non-zero coefficient. We observe that the vectors in $S_{t,k}$ have weight $2^t \cdot \binom{n-2t}{k-t}$. Regarding the question of minimum weights see also the paper by Frankl and Pach [1]. Rather surprisingly, in $E_{\lambda_{k,k}}$ we obtain:

PROPOSITION 4. *The set of minimum weight vectors in $E_{\lambda_{k,k}}$ is $S_{k,k}$.*

It appears that such a result should hold for arbitrary $t \leq k$. However, we were unable to prove this. Note that all the vectors $S_{t,k}$ have coefficients from the set $\{-1, 0, 1\}$ so that one may consider this question in arbitrary characteristic. The following argument holds independently.

PROOF. The proposition is obvious when $t = k = 1$. We proceed by induction. Thus let $x \neq 0$ be a minimum weight vector in $E_{\lambda_{k,k}}$ and denote its weight by $w(x)$. We know from above that

$$w(x) \leq 2^k. \tag{4}$$

Let Ω^* be the *support* of x; that is, the union of all sets appearing in x with non-zero coefficient, and let v be the cardinality of Ω^*. We can assume that $2k \leq v$, for otherwise the map ${}^-$ restricted to Ω^* is injective and $x = 0$.

Now let $\alpha \neq \beta$ be any points in Ω^* and write x, as in the proof of Proposition 2, uniquely as $x = \alpha \cup a + \beta \cup b + \{\alpha, \beta\} \cup c + d$. Since $0 = x^- = \alpha \cup (a^- + c) + \beta \cup (b^- + c) + \{\alpha, \beta\} \cup c^- + (a + b + d^-)$, we have in particular:

$$0 = a^- + c, \qquad 0 = b^- + c, \qquad 0 = c^- \qquad \text{and} \qquad a + b + d^- = 0. \tag{5}$$

Thus either $c = 0$ or, by induction, $w(c) \geq 2^{(k-2)}$. We now show that there must be at least one choice of α, β for which $c = 0$. For assume the contrary. Counting triples $(\alpha, \beta, \Delta \mid \alpha, \beta \in \Delta, \Delta$ appears in $x)$ we obtain $2^{(k-2)}v(v-1)/2 \leq w(x)k(k-1)/2$. This contradicts (4) above and the fact that $2k \leq v$.

Thus choose α and β such that $x = \alpha \cup a + \beta \cup b + d$. Notice that $a \neq 0 \neq b$ since α, β belong to Ω^*, but $a^- = 0 = b^-$ by (5). By induction on k we can assume that $2^{(k-1)} \leq w(a)$ and $2^{(k-1)} \leq w(b)$. Therefore $2^k \geq w(x) = w(a) + w(b) + w(d) \geq 2^k + w(d)$ implies that $d = 0$. Going back to (5) we see that $b = -a$, so that $x = (\alpha - \beta) \cup a$. This proves that $w(x) = 2^k$ and $w(a) = 2^{(k-1)}$. By induction, x belongs to $S_{k,k}$. □

5. Remarks

1. The symmetric group has rank $k+1$ as permutation group on the k-element subsets of Ω. This implies that the $E_{\lambda_{t,k}}$ are irreducible for each $t \leq k$ and so appear as Specht modules (see [2] or [3]). One can observe directly that $E_{\lambda_{t,t}}$ arises in Specht's construction from the tableau corresponding to the partition of type $(n-k, k)$. There are other situations in which the $E_{\lambda_{t,k}}$ are irreducible. These include, for instance, projective spaces over a finite field.

2. Often, the construction of combinatorial objects is equivalent to solving an equation of the type $x^{-r} = a$ in $F\mathbb{L}$, where x is required to be a vector of non-negative integer components. Thus, for instance, a t-design on Ω with block size k can be identified with a zero–one vector x in $F\mathbb{L}_k$ which satisfies $x^{-(k-t)} = a$, where a is a vector of constant entries. It seems desirable to obtain further invariants of the maps $^-$ or $^+$. This applies in particular to integer invariants such as elementary divisors and rational congruences. Some aspects of the latter are discussed in [6, Section 5].

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Johannes Siemons
School of Mathematics,
University of East Anglia,
Norwich NR4 7TJ, U.K.