On a problem of Wielandt and a question by Dembowski

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1. Introduction

Let G be a group with finite permutation representations on two sets Ω_1 and Ω_2. Let $\pi(g)$ and $\beta(g)$ be the number of points fixed by g in Ω_1 and Ω_2 respectively. Our principal assumption is $\pi(g) = \beta(g)$ for all g in G. Then clearly $|\Omega_1| = \pi(1) = \beta(1) = |\Omega_2|$ and both representations have the same kernel. Hence assume that they are both faithful. Furthermore, as the number of orbits on Ω_1 or Ω_2 is equal to the average number of points fixed by the group elements, G has the same number of orbits in both representations. Hence assume that they are transitive. Similar reasoning shows that G has the same permutation rank in both representations.

In the Kourovka Notebook (1983) Wielandt poses the following problem: If G acts primitively on Ω_2 does G always act primitively on Ω_1? There are many examples of inequivalent permutation representations with $\pi = \beta$, but none are known to contradict Wielandt's conjecture†. Generally it seems to be a hard problem and only little progress has been made towards a proof. The main result to date (see §2) is due to Förster and Kovács (1990) who show that Wielandt's problem can be reduced to the consideration of almost simple groups.

In Dembowski's book (page 212) a similar question appears in a more restricted situation: If G is a group of automorphisms of a finite projective plane, acting primitively on lines, is G then necessarily primitive on points? The answer to this question is now known, due to the classification of projective planes with primitive collineation group by Kantor (1987).

First we observe that Dembowski's problem is a subcase of Wielandt's as automorphisms of finite projective planes fix an equal number of points and lines. (In passing we note that this is *not* the case in infinite planes, see the paper of Mäurer 1988). As a proof one can quote Baer (1947) or alternatively an earlier result in Brauer (1941). There the "permutation lemma" states that if two permutation matrices A and B satisfy $AS = SB$ for some non-singular matrix S, then A and B represent similar permutations. In the case of a projective plane one can take S to be the incidence matrix of the plane.

By an incidence structure $\mathbb{S} = (\mathbb{P}, \mathbb{B}; \mathbb{I})$ we mean two disjoint and finite sets $\mathbb{P}$ and $\mathbb{B}$, the *point* set and *block* set of $\mathbb{S}$, with some incidence relation $\mathbb{I} \leq \mathbb{P} \times \mathbb{B}$. Dembowski's question now has an obvious generalisation: Let $\mathbb{S}$ belong to some class of incidence structures and let G be a group of automorphisms of $\mathbb{S}$. Does primitivity of G on $\mathbb{B}$ always imply primitivity of G on $\mathbb{P}$?

We need of course some general conditions on $\mathbb{S}$. The remark above concerning Brauer's permutation lemma suggests we might assume that the linear rank of an incidence matrix for $\mathbb{S}$ is equal to $|\mathbb{P}|$. This has the consequence that the associated permutation characters $\pi(g) = \mathrm{fix}_P(g)$ and $\beta(g) = \mathrm{fix}_B(g)$ satisfy $\pi \leq \beta$ so that $\beta = \pi + \chi$ where χ is some character of the automorphism group of $\mathbb{S}$. For details see theorem 3.2 in Camina and Siemons (1989).

Returning to Wielandt's problems one might therefore ask about groups G with faithful representations on sets Ω_1 and Ω_2, with (G, Ω_2) primitive and (G, Ω_1) imprimitive, such that $\pi \leq \beta$ for the corresponding permutation characters. In this situation we say that G is an *exception* with $\pi \leq \beta$. One can view Ω_1 and Ω_2 as the point and block sets of an incidence structure in which the incidence relation is given as some union of G-orbits on $\Omega_1 \times \Omega_2$. In section 3 we give examples of exceptions arising from the groups $PSL(2, p)$ and discuss the associated incidence structures.

2. Some observations and known results

Suppose that G acting faithfully on Ω_1 and Ω_2 is an exception with $\pi \leq \beta$. Let $P_1, \ldots, P_i, \ldots, P_t$ be a non trivial system of imprimitivity for the action of G on Ω_1. Let H be the subgroup fixing all the P_i sets. If $H \neq 1$ then it is transitive on Ω_2 as it is normal in G and it is also transitive on Ω_1 as $\pi \leq \beta$. Hence the action of fegree t is faithful and by choice we may assume it is primitive. Let π_0 denote the corresponding character. It is a simple matter to show that then $\pi_0 < \pi \leq \beta$. This gives first of all

Lemma 1. *Let (G, Ω_1, Ω_2) be an exception with $\pi \leq \beta$. Then every abelian subgroup acts intransitively on Ω_2.*

Proof. Let A be an abelian subgroup and consider the restriction of $\pi_0 < \pi \leq \beta$ to A. If A is transitive on Ω_2 then it must be transitive and faithful in all three representations. But then A is regular contradicting $\pi_0 < \pi$.

For exceptions with $\pi = \beta$ this observation is mentioned already in the Kourovka Notebook. The lemma implies in particular that exceptions can not have a solvable normal subgroup.

Theorem 2 (Förster and Kovács 1990). *An exception G with $\pi = \beta$ and minimal normal subgroup $T \times T \ldots \times T$, where T is a non-abelian simple group, gives rise to an exception $\underline{G}$ with $\underline{\pi} = \beta$ and $T \leq \underline{G} \leq Aut(T)$.*

Their proof shows a little more, namely that every almost simple exception conversely extends to exceptions with several simple factors. We expect that Theorem 2 carries through to the general situation $\pi \leq \beta$ but we have not yet completed a proof.

We now come to the combinatorial and geometric aspect of the problem and assume that additional requirements on the type of incidence structure is made. Delandtsheer and Doyen (1989) give an elegant and elementary proof for the following

Theorem 3. *If a 2-(v, k, λ) design with $v > (\binom{k}{2} - 1)^2$ has a block-transitive automorphism group then the group acts primitively on points.*

This paper is also a good source of reference to conditions on 2-designs which imply point-primitivity. It contains the conjecture that block-primitivity and $\lambda = 1$ should imply point-primitivity. This has been settled in Delandtsheer (1988, 1989) for small values of k and small permutation rank in the block representation and in Kantor (1973) for projective spaces of dimension at least 3.

Kantor (1987) characterizes finite projective planes with point-primitive collineation group. This work is based on a list of primitive permutation groups of odd degree and ultimately on the classification of finite simple groups. As a by-product, Dembowski's original question finally has an affirmative answer:

Theorem 4 (Kantor 1987). *Let Π be a projective plane of finite order q and let G be a group of automorphisms permuting the points primitively. Then either* (i): *Π is desarguesian and $G \geq PSL(3, q)$*

or (ii): *G is regular or a Frobenius group and the number of points of* Π *is a prime.*

In this context one should also mention the classification of affine planes with primitive collineation group by Hirame (1990).

Further evidence for the conjecture that block-primitivity implies point-primitivity in 2-(v, k, 1) designs can be derived from a result in Camina and Siemons (1989) which is independent of the classification theorem.

Theorem 5. *Let* D *be a* $2\text{-}(v, k, 1)$ *design with automorphism group* G *acting primitively on the blocks of* D. *Suppose that the stabilizer in* G *of two points* $p \neq p'$ *fixes all points in the block through* p *and* p'. *Then* G *acts primitively on the points of* D.

Proof. In Theorem 2 of Camina and Siemons (1989) we have shown that a block-*transitive* group as above either is flag-transitive or has odd order. In the first case G is primitive by a theorem of Higman and McLaughlin (1961). In the second case G is solvable and hence point primitive by Lemma 1.

3. Splitting designs

We construct examples of groups G with two faithful representations $(G, \mathbb{P})$ and $(G, \mathbb{B})$ such that the associated permutation characters satisfy $\pi \leq \beta$. The motivation is, of course, to try and find examples with $|\mathbb{P}| = |\mathbb{B}|$, contradicting Wielandt's conjecture. There has been no success so far nor have we been able to construct 2-designs with this property.

The construction can be described as follows: In $G = PSL(2, p)$ let G_0 be the one point stabilizer in the usual representation on the projective line $\mathbb{P}_0$ of $p + 1$ points with character $\pi_0 = 1 + \psi$ and let B be some maximal subgroup of G with induced character β. When $\mathbb{B}$ are the cosets of B in G, it is clear that $(\mathbb{P}_0, \mathbb{B})$ either is a 2-design and $\pi_0 \leq \beta$ or $(\mathbb{P}_0, \mathbb{B})$ is a trivial structure and π_0 and β have only the principle character in common. Now let P be a subgroup of G_0 with associated character π and let $\mathbb{P}$ be the cosets of P in G. This corresponds to splitting each point of the projective line into $|G_0 : P|$ *new points*. On the sets $\mathbb{P}$ and $\mathbb{B}$ incidence can be defined in various ways. Of course, it remains to be seen if $\pi \leq \beta$. For the characters of PSL we follow Dornhoff's (1971) notation.

Example 6. $B \cong A_4$. For p at least 7, the subgroup B is maximal if and only if $p \equiv 3$ or $7 \bmod 10$ and $p^2 \equiv 9 \bmod 80$. The cosets of B can be identified with sets of 4 harmonic points on the projective line and the resulting structure $(\mathbb{P}_0, \mathbb{B})$ is a 3-(p + 1, 4, 1) design. The character β can be calculated as $\beta = 1 + [(p + t_p)/12]\,\psi + [(p + t_p)/24]\,(\xi_1 + \xi_2) + \Sigma_k[(p + t_{pk})/12]\theta_k + \Sigma_l[(p + t_{pl})/12]\,\chi_l$ where the values for t_p, t_{pl} and t_{pk} are given in terms of various congruences on p, k and l.

Taking P as the subgroup of index 2 in G_0, hence $p \equiv 1 \bmod 4$, we find that $\pi = 1 + \psi + (\xi_1 + \xi_2)$ so that $\pi \leq \beta$ in all cases. Defining incidence between $\mathbb{P}$ and $\mathbb{B}$ suitably one obtains 2 - $(2(p + 1), 4, \Lambda)$ designs with $\Lambda = \{0, (p-1)/4\}$.

The smallest exception overall, in terms of group order, arises in this way when G is $PSL(2, 13)$ and $(\mathbb{P}, \mathbb{B})$ is a 2-(28, 4, {0, 3}) design. If S denotes its 28×91 incidence matrix, then SS^T has eigenvalues 52^1, 10^{13}, and $(13 \pm 3\sqrt{13})^7$, from which one can see that the multiplicities are the degrees of the characters $1, \psi, \xi_1$ and ξ_2.

Example 7. $B \cong D_{p-1}$. In this case B is the stabilizer of the 2-set $\{0,\infty\}$ on the projective line whose remaining two orbits are the squares and non-squares in the field. Choosing either, one obtains a 2-$(p + 1, (p-1)/2, (p-1)(p-3)/8)$ design. The corresponding character β can be calculated as $\beta = 1 + 2\,\psi + t_p(\xi_1 + \xi_2) + \Sigma_k\,\theta_k + 2\,\Sigma_{1 \leq i \leq (p-5)/8}\,\chi_i$ where $t_p = 1$ if $p \equiv 1 \bmod 8$ and 0 otherwise. As above, we let P be a subgroup of index 2 in G_0. Hence $\pi \leq \beta$ if $p \equiv 1 \bmod 8$.

Defining incidence between $\mathbb{P}$ and $\mathbb{B}$ suitably we now obtain 2-$(2(p + 1), (p-1)/2, \Lambda)$ designs with $\Lambda = \{0, (p-1)(p-5)/32, (p-1)^2/32\}$.

† *Note added in proof*: We mentioned Wielandt's problem to R Guralnick at the Durham Symposium. Later at the meeting he produced a counter example: it involves the triple cover of M_{22} in $PSL(45, 43)$. See his preprint note *Primitive Permutation Characters.*

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