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TACTICAL DECOMPOSITIONS IN FINITE INCIDENCE STRUCTURES

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TACTICAL DECOMPOSITIONS IN FINITE INCIDENCE STRUCTURES

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SUNTO. — Si studiano le decomposizioni tattiche in una struttura d'incidenza finita e gli spettri ad essa associati.

1. - Introduction.

Let P be a finite set of points, B a finite set of blocks and I an abstractly given incidence relation which tells us whether or not a given point is on a given block. The objective of this paper is to study such incidence structures employing concepts that are developed from the understanding of well known finite structures such as projective and affine space or undirected graphs etc. There one of the most fundamental ideas is similarity: the equivalence of elements in the struc-

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ture under its group of automorphisms. The notion of a tactical decomposition — due to Dembowski — abstracts from the properties of similarity and allows us to analyse a structure quite independently of its automorphism group. In some way this is the equivalence of elements and subsets being indistinguishable by means of counting incidence.

In an earlier paper [5] some of the elementary properties of tactical decompositions have been derived. For instance the fact that any partition of the elements in a structure has a unique (minimal) tactical refinement. This essentially is the process of partitioning elements according to colour schemes. They are the theme of chapters 2 and 3 in the present paper.

Linearization of a structure provides a natural setting for colour schemes and tactical decompositions: the elements are taken as basis vectors (over some field), thus spanning a point space and a block space. The incidence relation provides a pair of maps between these two spaces. As it turns out, a decomposition will be invariant under these incidence maps precisely when it is a tactical decomposition. Thus many techniques from linear algebra can be employed.

Certain subspaces of the point and block space are group invariant. They in turn give rise to group invariant decompositions, these are the kernels of a structure, defined in chapter 3. Among other results we obtain a bound in general for the number of distinct colour schemes arising from an arbitrary partition of the points (Theorem 3.3).

The compositions of the incidence maps are endomorphisms of the point and the block space. Their eigenvalues form the spectrum of a structure. Similarly the spectrum of a tactical decomposition may be defined. In theorem 4.4 we show that the spectrum of a tactical decomposition always is contained in the spectrum of the structure. Theorem 4.5 deals with the converse inclusion.

In the fourth chapter we also consider the case when the number of colour schemes takes its minimal value (i.e. the number of classes in the point partition). This leads to various characterizations of tactical decompositions.

This paper is a continuation of [5]. Despite this fact it is as self-contained as possible, with most of the notation given in chapter 2.

2. - Basic definitions and notation.

Let $S = (P, B)$ be a finite incidence structure with incidence relation I. The elements in P are called *points* and the elements in B are called *blocks*. If a point p is incident with a block b we denote this by pIb. Let P be a partition of P into r non-empty classes $P_1, \ldots, P_r$. Then $r = |P|$ denotes the number of classes while $|P_i|$ is the number of points in class P_i. When P' is a second partition of P we call P' a *refinement*, $P' \geq P$, of P if every class of P' is contained in some class of P. The same notation shall be used for the block set of S.

Let b be a block and let $c_{r'}(b)$ be the number of points in $P_{r'}$ that are incident with b. The vector $c(b) = (c_1(b), \ldots, c_{r'}(b), \ldots, c_r(b))$ is the *colour scheme* of b. The relation $b \sim b'$ if and only if $c(b) = c(b')$ defines a partition of B into $\sim$equivalence classes. This we call the *colour scheme partition* of B relative to P. It shall be denoted by B_P. Let $B_1, \ldots, B_s$ be the classes of B_P and choose a $b_{s'}$, in $B_{s'}$ for every $1 \leq s' \leq s$. The $(r \times s)$ matrix

$$C(P) = \begin{bmatrix} c(b_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c(b_s) \end{bmatrix}$$

is the *colour scheme matrix* of P.

An automorphism of S is a pair $g = (g_P, g_B)$ where g_P and g_B are permutations of the point and the block set, respectively, such that pIb iff $g_P(p) I g_B(b)$. The images of p and b are also denoted by p^g and b^g, similarly $(P_r)^g$ denotes the set of all images of points in P_r etc. The group of all automorphisms of S is denoted by AUT (S). If $G \leq$ AUT (S) and if P is a partition of P we say that P is G-*invariant* if $(P_{r'})^g$ is a class of P for every g in G and every $1 \leq r' \leq r$. If $(P_{r'})^g = P_{r'}$ for all g in G and all $1 \leq r' \leq r$, we say that P is G-*fixed*.

LEMMA 2.1. - *Suppose that* P *is* G-*invariant* (G-*fixed*). *Then* B_P *is* G-*invariant* (G-*fixed*).

PROOF. - This follows immediately from the definition. □

It will be extremely useful to describe a structure (P, B) in terms of vector spaces and incidence maps. Here I refer in particular to [5]. When R is a field of characteristic zero the *point space* $\mathrm{R}P$ is the vector space of all formal sums $\sum \mathrm{r_i} \cdot p_\mathrm{i}$ where $\mathrm{r_i}$ belongs to R and p_i to P. An inner product $\langle\ ,\ \rangle$ is defined through the δ-function $\langle p_\mathrm{i}, p_\mathrm{j}\rangle = \delta_\mathrm{ij}$ when p_i and p_j are points in P. For a partition class $\mathrm{P_i}$ the vector $\mathrm{p_i}$ shall denote the sum over all points in $\mathrm{P_i}$. Thus for instance $\langle p, \mathrm{p_i}\rangle = 1$ if and only if p belongs to $\mathrm{P_i}$, note also that $\langle \mathrm{p_i}, \mathrm{p_i}\rangle = |\mathrm{P_i}|$.

The space generated by $\mathrm{p_1}, \ldots, \mathrm{p_i}, \ldots, \mathrm{p_r}$ where $\mathrm{P_1}, \ldots, \mathrm{P_r}$ are all the classes of the partition P is called the *partition space* associated to P and is denoted by RP. In general, a subspace U of $\mathrm{R}P$ is called a partition space if $\mathrm{U} = \mathrm{RP}$ for some partition P. In § 2 of [5] I have shown that any subset W of RP is contained in a unique partition space of minimal dimension. This space is denoted by L(W) and the corresponding partition is denoted by $\mathrm{P_W}$.

Precisely the same notions shall hold for the block set: $\mathrm{R}B$ denotes the block space, $\langle b_\mathrm{i}, b_\mathrm{j}\rangle = \delta_\mathrm{ij}$ is the inner product and RB is the partition space associated to a partition B of the block set.

The incidence relation leads to incidence maps $\partial^+ : \mathrm{R}P \longrightarrow \mathrm{R}B$ and $\partial^- : \mathrm{R}B \longrightarrow \mathrm{R}P$ given by $\partial^+(p) = \sum_{pIb} b$ and $\partial^-(b) = \sum_{pIb} p$. These maps are adjoint to each other: pIb is equivalent to $\langle \partial^-(b), p\rangle = 1$ but also to $\langle \partial^+(p), b\rangle = 1$. Therefore $\langle \partial^+(\mathrm{u}), \mathrm{w}\rangle = \langle \mathrm{u}, \partial^-(\mathrm{w})\rangle$ whenever u belongs to $\mathrm{R}P$ and w to $\mathrm{R}B$.

The components of the colour scheme vectors now are easily expressed in terms of the incidence map: $c_{\mathrm{r}'}(b) = \langle \partial^-(b), \mathrm{p_{r'}}\rangle = \langle b, \partial^+(\mathrm{p_{r'}})\rangle$. When $\mathrm{B_1}, \ldots, \mathrm{B_{s'}}, \ldots, \mathrm{B_s}$ are the classes of $\mathrm{B_P}$ and $b_{\mathrm{s}'}$ is a block belonging to $B_{\mathrm{s}'}$, we calculate $\partial^+(\mathrm{p_{r'}}) = \sum_b \langle \partial^+(\mathrm{p_{r'}}), b\rangle \cdot b = \sum_b \langle p_{\mathrm{r}'}, \partial^-(b)\rangle \cdot b = \sum_{\mathrm{s}'} \mathrm{c_{r'}}(b_{\mathrm{s}'}) \cdot \sum_{b\in \mathrm{B_{s'}}} b = \sum_{\mathrm{s}'} \mathrm{c_{r'}}(b_{\mathrm{s}'}) \cdot \mathrm{b_{s'}}$.

This shows that $\partial^+(\mathrm{RP}) \leq \mathrm{RB_P}$ and that C (P) is the matrix of ∂^+ (with respect to the bases $\mathrm{p_1}, \ldots, \mathrm{p_r}$ and $\mathrm{b_1}, \ldots, \mathrm{b_s}$) when ∂^+ is restricted to RP.

Interchanging the rôle of points and blocks, starting from an arbitrary partition B of the block set we define the colour scheme partition $\mathrm{P_B}$ of the point set. This yields colour scheme vectors $\mathrm{c}\,(p)$ in which the components count the number of blocks through p in each block class. In the same way the colour scheme matrix C (B) is defined. By the argument above $\partial^-(\mathrm{RB}) \leq \mathrm{RP}_\mathrm{B}$ and C (B) is the matrix of the restricted map $\partial^- : \mathrm{RB} \longrightarrow \mathrm{RP}_\mathrm{B}$.

3. - Some group invariant partitions.

Let K^+ be the kernel of ∂^+ and let K^- be the kernel of ∂^-. As we have seen in the last chapter these subspaces define a partition $\mathrm{P}_{\mathrm{K}^+}$, the *point kernel* and B_{K^-}, the *block kernel* of the structure. They shall be abbreviated by P^0 and B^0. So, for instance, P^0 is the trivial partition into a single class if and only if K^+ is the null space, i.e. ∂^+ is injective. This is a most common situation for interesting incidence structures. In bipartite, connected graphs, however, K^+ is spanned by the vector $p_1 - p_2$ where p_i is the sum of all vertices in a part of the graph, $i = 1, 2$. Thus P^0 consist of two classes, and they are the parts of the graph. (See also [6]). As an example of a block kernel we may consider the complete graph on 4 vertices. Here B^0 consist of 3 classes, each being a pair of edges with no vertex in common.

PROPOSITION 3.1. - P^0 *and* B^0 *are invariant under all automorphisms of* S. *If* AUT (S) *is transitive on* P (*on* B) *then* $|\mathrm{P}^0|$ *divides* $|P|$ ($|\mathrm{B}^0|$ *divides* $|B|$). *If* AUT (S) *acts primitively on* P (*on* B) *then* P^0 (B^0) *is a trivial partition.*

Thus, for primitive point action K^+ either is the null space or otherwise is sufficiently large to generate the whole of $\mathrm{R}P$ as the smallest partition space containing it.

PROOF. - When $g = (g_P, g_B)$ is an automorphism of S, then the action on points and blocks naturally extend to an action of g on the point and block spaces as a linear transformation. It follows very easily that $\partial^+ \cdot g_P = g_B \cdot \partial^+$ and as a consequence $\partial^+(w) = 0$ implies $\partial^+(w^g) = 0$. This shows that K^+ is invariant under all automorphisms, hence also $\mathrm{P}_{\mathrm{K}^+} = \mathrm{P}^0$.

If AUT (S) acts transitively on P, the classes of P^0 are blocks of imprimitivity, thus their number and their size divide the number of points. When AUT (S) acts even primitively a class of P^0 must either be a single point or consist of all the points, thus P^0 is a trivial partition. The arguments for the block kernel are precisely the same. □

When P' is a subset of points in a structure (P, B) we consider the *substructure* (P', B) in which incidence is defined as in (P, B). The following theorem shows that by removing a point in each but one of the classes of P^0 one can always achieve a substructure in which the point kernel is trivial.

THEOREM 3.2. - *Let* $P^0 = \{P_1^0, \ldots, P_i^0, \ldots,\}$ *be the point kernel of a structure* (P, B) *and let* P' *be a subset of* P *such that* $P_i^0 \cap P' = P_i^0$ *for at most one of the classes of* P^0. *Then* $\partial^+ : RP' \longrightarrow RB$ *is injective. In particular* $|P'| \leq |B|$.

PROOF. - Suppose that $a = \sum a_i p_i$ belongs to the kernel of $\partial^+ : RP' \longrightarrow RB$, that is $a_i = 0$ if $p_i \notin P'$ and $\partial^+(a) = 0$. As $K^+ \leq RP^0$, also $a = \sum_j a_j \cdot p_j^0$ and from the condition on $P_j^0 \cap P'$ it follows that $a = 0$ or $a = a_1 \cdot p_1^0$ (when P_1^0 is the class entirely contained in P'). But then $\partial^+(a) = 0$ implies that a_1 must be zero. Thus ∂^+ is injective on RP'. □

The *meet* $P \wedge P'$ of two partitions of the point set is the partition $\bar{P}$ of largest class number satisfying $P \geq \bar{P}$ and $P' \geq \bar{P}$. Next we give a bound for the distinct colour schemes arising from an arbitrary point partition.

THEOREM 3.3. - *Let* P *be a partition of the point set of a structure* (P, B) *with point kernel* P^0. *Then the number of classes of the induced colour scheme partition* B_P *satisfies* $|B_P| \geq |P| - |P \wedge P^0| + 1$.

PROOF. - We have seen that ∂^+ maps RP into RB_P. As the dimensions of these spaces are the class numbers, we obtain $|B_P| \geq |P| - \dim(K^+ \cap RP)$. Thus we are lead to consider $K^+ \cap RP$. If a is an element in this space then a also belongs to RP^0. Thus a can be written in two ways: $a = \sum_i a_i \cdot p_i$ and $a = \sum_j a_j \cdot p_j^0$. This implies that classes P_i and P_j^0 with non-empty intersection carry the same coefficient, but this is the same as saying that a belongs to $R(P \wedge P^0)$. Finally observe that a standard basis vector in this space (with coefficients 0 or 1) does not lie in K^+. Therefore $|P \wedge P^0| - 1 \geq \dim(K^+ \cap RP)$ and the result is proved. □

4. - Tactical decompositions.

In this chapter let (P, B) be a fixed incidence structure in which $|P| \leq |B|$. When P and B are a pair of given partitions we may compare them to their respective induced colour scheme partitions. This leads to the following

DEFINITION. - (P, B) is $^+$*tactical* if $B \geq B_P$. (P, B) is $^-$*tactical* if $P \geq P_B$. If both properties are satisfied then (P, B) is a *tactical pair.*

In the literature also the terms block tactical and point tactical are used, see for instance [3]. As we have seen in § 2 the colour scheme partitions satisfy $\partial^+(RP) \leq RB_P$ and $\partial^-(RB) \leq RP_B$. From proposition 2.1 in [5] it follows that (P, B) is $^+$tactical ($^-$tactical) if and only if $\partial^+(RP) \leq RB$ $(\partial^-(RB) \leq RP)$. Theorem 3.3 now gives a general version of Block's Lemma (see page 21 in [4]):

THEOREM 4.1. - *If* (B, P) *is a* $^+$*tactical pair then* $|B| \geq |P| - |P \wedge P^0| + 1$.

Tactical pairs occur naturally as the orbit partitions of an automorphism group of a structure. More generally a pair is tactical if certain regularity conditions are met. This can be seen for instance in the case of designs. Let $S = (\binom{X}{t}, \binom{X}{k})$ where $\binom{X}{j}$ denotes the collection of all j-element subsets of the finite set X. When $t \leq k$ let incidence be given by set inclusion. Let P be the partition of $\binom{X}{t}$ into a single class and suppose that $B = \{B_1, ..., B_i, ...,\}$ is a partition of $\binom{X}{k}$. It now follows that (P, B) is tactical if and only if (X, B_i) is a t-design for every class of B.

4.1. - Spectra.

Let ν_P be the map $\partial^-\bullet\partial^+ : \mathrm{R}P \longrightarrow \mathrm{R}P$ and ν_B the map $\partial^+\bullet\partial^-$: $\mathrm{R}B \longrightarrow \mathrm{R}B$. As the incidence maps are adjoint to each other, the maps ν_P and ν_B are symmetric and so their eigenvalues are real.

The *spectrum* of S, $\mathrm{Spec}\,(P, B)$, is the collection of eigenvalues, with multiplicities, of the transformation ν_P. First we show that ν_P and ν_B essentially have the same eigenvalues.

THEOREM 4.2. - *If* $|P| \leq |B|$ *then* $\mathrm{Spec}\,(P, B)$ *is obtained by removing* $|B| - |P|$ *zeros from the sequence of eigenvalues of* ν_B.

PROOF. - The matrices of ν_B and ν_P are $\mathrm{I}\bullet\mathrm{I}^{\mathrm{T}}$ and $\mathrm{I}^{\mathrm{T}}\bullet\mathrm{I}$ where I is some incidence matrix of S. Thus we consider the characteristic polynomials $\det(\mathrm{t} - \mathrm{I}\bullet\mathrm{I}^{\mathrm{T}})$ and $\det(\mathrm{t} - \mathrm{I}^{\mathrm{T}}\bullet\mathrm{I})$. The theorem now follows from

LEMMA 4.3. - *Let* A *be an* $(\mathrm{r} \times \mathrm{s})$*-matrix and* B *an* $(\mathrm{s} \times \mathrm{r})$*-matrix, both real, such that* $\mathrm{r} \leq \mathrm{s}$. *Then* $\det(\mathrm{t} - \mathrm{BA}) = \mathrm{t}^{\mathrm{s}-\mathrm{r}} \bullet \det(\mathrm{t} - \mathrm{AB})$.

PROOF. - First let A and B be square matrices and suppose that A is invertible. Then, as $(1 - \mathrm{AB})$ is conjugate to $(1 - \mathrm{BA})$ we have $\det(1 - \mathrm{AB}) = \det(1 - \mathrm{BA})$. Consider this as a polynomial identity for the real entries of A. By density and continuity arguments the equation then also holds when A is singular. Using this equation for $\mathrm{A}' = \mathrm{t}\bullet\mathrm{A}$ we obtain $\mathrm{t}^{\mathrm{r}} \bullet \det(1 - \mathrm{BA}) = \det(\mathrm{t} - \mathrm{BA}') = \det(\mathrm{t} - \mathrm{A}'\mathrm{B})$ and the lemma holds for square matrices. For rectangular matrices we augment A by rows of zeros to obtain a square matrix A^0 and B by columns of zeros to obtain a square matrix B^0. The result now follows from the consideration of $\mathrm{A}^0\mathrm{B}^0$ and $\mathrm{B}^0\mathrm{A}^0$. □

When (P, B) is a tactical pair then ν_{P} can be restricted to $\nu_{\mathrm{P}} = \partial^-\bullet\partial^+ : \mathrm{RP} \longrightarrow \partial^+(\mathrm{RP}) \leq \mathrm{RB} \longrightarrow \partial^-(\mathrm{RB}) \leq \mathrm{RP}$. The *spectrum* of (P, B), denoted Spec (P, B), is the collection of eigenvalues of ν_{P}, with multiplicities.

THEOREM 4.4. - *Let* (P, B) *and* (P′, B′) *be tactical pairs in a structure* (P, B) *such that* $\mathrm{P} \geq \mathrm{P}'$. *Then the spectrum of* (P, B) *is a subsequence the spectrum of* (P′, B′). *In particular, the spectrum of* (P, B) *is a subsequence of* $\mathrm{Spec}\,(P, B)$.

PROOF. - From $P \geq P'$ it follows that $RP \geq RP'$ and so $\nu_{P'}$ is the restriction of ν_P to an invariant subspace. This implies that eigenvalues of $\nu_{P'}$ are eigenvalues of ν_P. □

Let (P, B) and (P′, B′) be tactical pairs and suppose that some permutation g of the point set maps the classes of P onto the classes of P′. Viewing g as a linear map we obtain a transformation $g: RP \longrightarrow RP'$. Then g is a *similarity* between P and P′ if $g^{-1} \cdot \nu_{P'} \cdot g = \nu_P$ as a map of RP and we say that P is *similar* to P′ if there is a similarity of this kind. Clearly tactical pairs with similar point partitions have the same spectrum. When p is a point of P then there is a unique tactical pair (P^p, B^p) of minimal class number such that $\{p\}$ is a class of P^p, see theorem 2.3 in [5]. By the previous theorem Spec (P^p, B^p) is contained in Spec (P, B). We now deal with the converse:

THEOREM 4.5. - *For every eigenvalue* λ *in* Spec (P, B) *there is a point* p *such that* λ *also belongs to* Spec (P^p, B^p). *If* $P^{p'}$ *is similar to* P^p *for every point* p' *in* P *then* Spec (P, B) *and* Spec $(P^{p'}, B^{p'})$, *as sets, are equal and the number of distinct eigenvalues is a lower bound for* $|P^p|$ *and* $|B^p| - 1$, *the class numbers of this pair.*

The assumption of similarity holds in particular when AUT (S) is point transitive. When all eigenvalues in Spec (S) are distinct one shows easily that AUT (S) is an elementary abelian 2-group and, independently, that point similarity implies that each P^p consists of single point classes.

PROOF. - Let x be an eigenvector for λ. We may assume that $x = p_1 + \sum_{i \geq 2} a_i \cdot p_i$. Let $P_1 = \{p_1\}, \ldots, P_i, \ldots$ be the classes of P^p for $p = p_1$. Now define $y = \sum_i \langle x, p_i \rangle \langle p_i, p_i \rangle^{-1} \cdot p_i$, an element in RP^p. Then $\langle y, p_j \rangle = \langle x, p_j \rangle$ for all p_j so that $y \neq 0$ as $\langle y, p_1 \rangle = 1$. As ν_P restricts to RP^p, each $\nu_P(p_j)$ is a linear combination of the p_i's so that also $\langle y, \nu_P(p_j) \rangle = \langle x, \nu_P(p_j) \rangle$ for all j. This leads to $\langle \nu_P(y), p_j \rangle = \langle y, \nu_P(p_j) \rangle = \langle x, \nu_P(p_j) \rangle = \langle \nu_P(x), p_j \rangle = \langle \lambda \cdot x, p_j \rangle = \langle \lambda \cdot y, p_j \rangle$ for all j. But this means that y is an eigenvector in RP^p with eigenvalue λ.

Similarity, as we remarked above, implies equality of spectra so that every eigenvalue in Spec (P, B) belongs to Spec (P^p, B^p) for a fixed p. As $|P^p|$ is the dimension d of the space RP^p, the number of distinct eigenvalues in Spec (P, B) is a lower bound for d. The bound for the number of classes in B^p now follows from lemma 4.3. □

THEOREM 4.6. - *Let* P *be a partition of the point set in a structure* (P, B).

i) *There is a partition* B *of* B *such that* (P, B) *is a tactical pair if and only if* $P_{(B_P)} \leq P$.

ii) *For a tactical pair* (P, B) *in which zero does not belong to its spectrum we have* $P_{(B_P)} = P$ *and* Spec (P, B) *is the set of eigenvalues of* $C(B_P) \cdot C(P)$. *In particular* $|B| \geq |P|$.

PROOF. - i) Suppose (P, B) is tactical. Then $RB_P \leq RB$ implies $\partial^-(RB_P) \leq \partial^-(RB) \leq RP$. From the construction of $P_{(B_P)}$ it follows that $RP_{(B_P)}$ is the smallest partition space containing $\partial^-(RB_P)$. Thus $RP_{(B_P)} \leq$ $\leq RP$ and so $P_{(B_P)} \leq P$. Now suppose the latter holds. Then $B = B_P$ yields a tactical pair (P, B).

ii) From $\partial^- \bullet \partial^+ : P \longrightarrow RB_P \longrightarrow RP_{(B_P)} \leq RP$ and the fact that ν_P is non-singular we conclude that $P_{(B_P)} = P$. With respect to the standard bases these maps have matrices C (P) and $C(B_P)$. Thus the spectrum of (P, B) are the eigenvalues of $C(B_P) \cdot C(P)$. □

4.2. - Characterizations by flags.

A *flag* is a pair (p, b) such that p is incident with b. When $P_1, \ldots, P_i, \ldots$ are the classes of a partition and p is a point we count the number of flags simultaneously through p and a point of P_i: let $d_i(p) = |\{(p', b) \mid p'Ib,\ p' \text{ in } P_i \text{ and } pIb\}|$.

In terms of the incidence maps $d_i(p)$ is easily seen to be equal to $\langle \partial^+(p_i),\ \partial^+(p) \rangle$.

THEOREM 4.7. - *Suppose that* (P, B) *is a pair of partition of the point and the block set of a structure* (P, B).

i) *If* (P, B) *is tactical, then* (*) : $d_i(p) = d_i(p')$ *whenever* p *and* p' *belong to the same class of* P, *for all* i.

ii) *If* 0 *does not belong to* Spec (P, B) *and if* $|P| = |B_P|$, *then* (*) *implies that* (P, B_P) *is tactical.*

PROOF. - From the orthogonality relations among the points p and partition vectors p_i we see that w belongs to RP if and only if $\langle w, p\rangle = \langle w, p_j\rangle \cdot \langle p_j, p_j\rangle^{-1}$ for all j and all p in P_j. If (P, B) is tactical, then $w := \partial^- \cdot \partial^+ (p_i)$ belongs to RP for every i. Thus if p and p' are in P_j then $\langle \partial^- \cdot \partial^+ (p_i), p\rangle = \langle \partial^+ (p_i), \partial^+ (p)\rangle = d_i(p) = \langle \partial^- \cdot \partial^+ (p_i), p_j\rangle \cdot \langle p_j, p_j\rangle^{-1} = \langle \partial^- \cdot \partial^+ (p_i), p'\rangle = d_i(p')$. This proves the first statement.

Now we suppose that $0 \notin \mathrm{Spec}(P, B)$. Then $\partial^+ : RP \longrightarrow RB_P$ is an injection and as the dimension of both spaces is $|P| = |B_P|$, ∂^+ is even a bijection. Thus $\{\partial^+(p_i) \mid i = 1, ...\}$ forms a basis of RB_P. As we have seen before, (P, B_P) forms a tactical pair precisely if $\partial^-(RB_P) \leq \leq RP$. Thus it suffices to show that $\partial^- \cdot \partial^+ (p_i)$ belongs to RP. As in the first part of the proof this will be the case precisely if (*) is satisfied. □

In certain types of structures condition (*) automatically holds. This is the case for instance in a t-design, $t \geq 2$: $d_i(p)$ only depends on the size of the class P_i. As a consequence of Fisher's inequality $0 \notin \mathrm{Spec}(P, B)$. This gives

COROLLARY 4.8. - *Suppose that* (P, B) *is a t-design,* $t \geq 2$, *and let* P *be a partition of the point set such that* $|P| = |B_P|$. *Than* (P, B_P) *is a tactical pair.*

Note that this is proposition 2.2 in [3].

A structure is a *linear space* if for any two points $p \neq p'$ there is precisely one block incident with both p and p' and this block is not incident with every point of the structure. See for instance [1]. There it is also shown that $0 \notin \mathrm{Spec}(P, B)$. Again condition (*) is satisfied for arbitrary partitions.

COROLLARY 4.9. - *Suppose that* (P, B) *is a linear space and let* P *be a partition of the point set such that* $|P| = |B_P|$. *Then* (P, B_P) *is a tactical pair.*

In complete contrast condition (*) sometimes entirely characterizes tactical pairs. This is the situation in graphs. Let Γ be a finite, undirected graph that is connected. Let P denote the vertex set and B the edge set. When $P = \{P_1, ..., P_i, ...\}$ is a partition of the vertices we consider the subgraphs Γ_{ij} whose vertices are $P_i \cup P_j$ and whose edges link vertices in P_i to vertices in P_j. Thus $d_i(p)$ is the degree in Γ_{ij} of a vertex p belonging to P_j. We say that a bipartite graph is *biregular* if the vertex degree in each part is constant. The condition (*) now is equivalent to: Γ_{ij} is biregular and Γ_{ii} is regular for all $i \neq j$.

THEOREM 4.10. - *Let* $\Gamma = (P, B)$ *be a finite graph and* $P = \{P_1, \ldots, P_i, \ldots\}$ *a partition of the vertex set. Then* (P, B_P) *is tactical if and only if* Γ_{ij} *is biregular and* Γ_{ii} *is regular for all* $i \neq j$.

PROOF. - By the above remarks and theorem 4.6 we only need to show that the regularity conditions imply that (P, B_P) is tactical. First we note that the classes in B_P are just the edges in the Γ_{ij}, thus we shall denote them by B_{ij}. We have to show that $\partial^-(b_{ij})$ belongs to $RP : \langle \partial^-(b_{ij}), p \rangle = \langle b_{ij}, \partial^+(p) \rangle = d_i(p)$ (if p belongs to P_j) or $= 0$ (if p does not belong to P_i or P_j). In any case this implies that $\langle \partial^-(b_{ij}), p \rangle = \langle \partial^-(b_{ij}), p' \rangle$ whenever p and p' are vertices in the same partition class. This completes the proof. □

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