Recognising direct products from their conjugate type vectors

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Dedicated to Helmut Wielandt on his 90th birthday

1 Introduction

In two long and interesting articles Mark L. Lewis, [7, 8], considered problems of the relation between the structure of a finite group G and $\mathrm{cd}(G)$, the set of the degrees of irreducible characters of G. In the second article the following two theorems are proved.

Theorem A Let G be a finite group with $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$ where p, q and r are distinct primes. Then $G = A \times B$ where $\mathrm{cd}(A) = \{1, p\}$ and $\mathrm{cd}(B) = \{1, q, r\}$.

Theorem B Let G be a finite group with $\mathrm{cd}(G) = \{1, p, q, r, s, pr, ps, qr, qs\}$ where p, q, r and s are distinct primes. Then $G = A \times B$ where $\mathrm{cd}(A) = \{1, p, q\}$ and $\mathrm{cd}(B) = \{1, r, s\}$.

*Supported by an EPSRC postdoctoral grant.

He also gives an example to show that if $\mathrm{cd}(G) = \{1, p, q, pq\}$, where p and q are distinct primes, then G is not necessarily a direct product.

In this paper we consider analogous problems for the set of sizes of conjugacy classes of G. See [5] for results of a similar nature. There is a strong relation between information about character degrees and sizes of conjugacy classes. If the multiplication constants for the conjugacy classes is known then the character table can be reconstructed and similarly in reverse. However, if one knows only the sizes then there is less complete information and it is not possible to obtain a complete translation. To illustrate this point, if the conjugacy classes have sizes $\{1, p, q, pq\}$ then the group is a direct product, [3, Theorem 2], in contrast to Lewis' example.

The study of the structure of a group given information about its conjugacy class sizes has a long history, for example in 1953 Baer considered such a problem [1]. In that paper he gave an unpublished result of H. Wielandt which is reproduced in the next section. The authors, in [4], generalised both Baer and Wielandt's results. In this paper we use those ideas to state and prove results which are analogues of Lewis' results, our Theorems 1 and 2. We note that these results are both easier and stronger than those of Lewis. In no case do we need to restrict to conjugacy classes of square-free size. Also, in the first theorem we extend the number of primes involved and in the second the number of factors. Finally, we do not use the classification of finite simple groups in the proofs.

2 Definitions & Notation

Throughout this paper G denotes a finite group. If $x \in G$ we denote the conjugacy class of x in G by x^G. Note that $|x^G| = [G : C_G(x)]$ where $C_G(x)$ denotes the centraliser of x in G and thus $|x^G|$ is called the *index of x in G*. N. Itô introduced the following definition in [6].

Definition Let $n_r > \cdots > n_2 > n_1 = 1$ denote the distinct indices of elements of a finite group G. Then $(n_r, \ldots, n_2, n_1)$ is called the *conjugate type vector* of G.

To deduce results about G given its conjugate type vector is an ongoing quest. Note that in considering such questions abelian direct factors are ignored. In

[4] the authors introduced the *product of conjugate type vectors*:

$$(n_r, \ldots, n_2, n_1) \times (m_s, \ldots, m_2, m_1)$$

is the ordered set $\{n_i m_j \mid 1 \leq i \leq r,\ 1 \leq j \leq s\}$. The point being that if H and G are finite groups then the conjugate type vector of $H \times G$ is the product of the conjugate type vectors of H and G.

R. Baer characterised the following groups in [1].

Definition A finite group G is called a *Baer group* if all elements of prime-power order have prime-power index.

In [4] the authors considered the following groups.

Definition Let G be a finite group and q a prime. G is a *q-Baer* group if all q-elements have prime-power index.

Amongst other things the authors proved that if G is a q-Baer group then there exists a prime p such that all q-elements have p-power index. In proving this they generalised the following well-known Lemma of Wielandt [1, Lemma 6].

Wielandt's Lemma. Let G be a finite group. If $x \in G$ is a p-element of p-power index for a prime p, then $x \in \mathrm{O}_p(G)$.

Generalisation of Wielandt's Lemma.[4, Proposition 1]
Let G be a finite group and p a prime. Suppose $x \in G$ has p-power index, then $[x^G, x^G] \subseteq \mathrm{O}_p(G)$.

In [2] the following definition is introduced.

Definition Let $C \subseteq G$ a finite group. Then

$$\ker(C) = \{x \in G \mid Cx = C\} \leq G.$$

3 Proofs

Lemma 1 *Let G be a finite group such that p^a is the highest power of the prime p which divides the index of an element of G. Assume that there exists a p-element of index p^a in G. Suppose m is the index of an element of G*

such that $(m, p) = 1$. Then there exists a p'-element, say y, of G of index m such that xy has index $p^a m$.

Proof. Suppose x is a p-element of index p^a. Then, by [3, Theorem 1], there exists a normal p-complement K. Furthermore, $K \leq C_G(x)$. Let u be a p'-element of $C_G(x)$ then u has index prime to p in $C_G(x)$ since $C_G(xu) = C_G(x) \cap C_G(u)$. So, by [3, Lemma 1], the Sylow p-subgroup P_x of $C_G(x)$ is a direct factor of $C_G(x)$ and $C_G(x) = P_x \times K$. Let y be an element of index m, then $y \in C_G(x)$, since y centralises $\mathrm{O}_p(G)$ and $x \in \mathrm{O}_p(G)$ by Wielandt's Lemma. We may assume $y \in K$ and thus xy has index $p^a m$ in G, as required.□

Lemma 2 *Let $x, y \in G$ a finite group. Suppose $|x^G| = p^a$ and $|y^G| = q^b$ where p and q are distinct primes and $p^a < q^b$. Also suppose there does not exist a conjugacy class of G of order divisible by pq. Then x is a q-element (up to multiplication by central elements).*

Proof. Let $x = x_1 x_2$ where x_1 is an r-element for some prime r and x_2 has order coprime to r. Note that both x_1 and x_2 have index a power of p which is smaller than q^b. Suppose $|x_1^G| \neq 1$.

Let B denote x_1^G and C denote y^G. Since $(|B|, |C|) = 1$ it follows that $CB = D$ a conjugacy class of G. Clearly $|D| \geq |C|$ and also $|D|$ divides $|C||B|$. So, by the hypothesis of the lemma, $|D| = |C|$. We repeat the argument and see that DB^{-1} is a conjugacy class of G. Also $C \subseteq CBB^{-1} = DB^{-1}$, so that $C = CBB^{-1}$. Thus $H = \langle BB^{-1} \rangle \leq \ker(C)$ and it follows that $|H|$ divides $|C|$, i.e. $|H|$ is a power of q. However, by the generalisation of Wielandt's Lemma, $\langle BB^{-1} \rangle \subseteq \mathrm{O}_{p,r}(G)$, this contradicts the previous statement unless $r = q$. It follows that x_2 is in the centre of G. □

Theorem 1 *Suppose G has conjugate type vector*

$$(p_s^{a_s}, \ldots, p_1^{a_1}, 1) \times (q_r^{b_r}, \ldots, q_1^{b_1}, 1)$$

where $p_1, \ldots, p_s, q_1, \ldots, q_r$ are distinct primes. Then $r, s \leq 2$ and $G = A \times B$ where A has conjugate type vector $(p_s^{a_s}, \ldots, p_1^{a_1}, 1)$ and B has conjugate type vector $(q_r^{b_r}, \ldots, q_1^{b_1}, 1)$.

Proof. If $r = s = 1$ this is [3, Theorem 2]. So assume $s > 1$. Let $x, y_i \in G$ with $|x^G| = p_1^{a_1}$ and $|y_i^G| = p_i^{a_i}$ for $2 \leq i \leq s$. Then, by Lemma 2, x is a p_i-element for each i. Thus $s = 2$ and x is a p_2-element. In $C_G(x)$ all $p_2^{'}$-elements have index prime to p_2 so $C_G(x) = P_2 \times L$ where P_2 is a Sylow p_2-subgroup of G by [3, Lemma 1]. Using Lemma 1, it follows that all p_2-elements have index $p_1^{a_1}$ or are central. Also, any q_i-element has a conjugate in L and thus has index prime to p_2. Further, an element of index $p_2^{a_2}$ must be a p_1-element, call such an element y. Then, similarly, $C_G(y) = P_1 \times M$ where P_1 is a Sylow p_1-subgroup. Thus all p_1-elements have p_2-index and all q_i-elements have index prime to p_1.

Thus we have shown that if $s \geq 2$ then $s = 2$ and all elements of prime-power order have prime-power index. So G is a Baer group and the result follows from [1, Theorem p.27].$\Box$

The following theorem is proved similarly.

Theorem 2 *Suppose G has conjugate type vector*

$$(p_{1,n_1}^{a_{1,n_1}}, \ldots, p_{1,1}^{a_{1,1}}, 1) \times \cdots \times (p_{r,n_r}^{a_{r,n_r}}, \ldots, p_{r,1}^{a_{r,1}}, 1)$$

where $p_{i,j}$ are distinct primes and $n_k \geq 2$ for $1 \leq k \leq r$. Then $n_k = 2$ for all k and $G = A_1 \times \cdots \times A_r$ where A_i has conjugate type vector $(p_{i,2}^{a_{i,2}}, p_{i,1}^{a_{i,1}}, 1)$.

Proof. We prove this theorem by induction. The case $r = 1$ follows from [1, Theorem p.27] and the case $r = 2$ is covered in the previous theorem. Suppose $r > 1$ and the result holds for smaller r, we find the required A_1 and the result will follow by induction.

Suppose $|x^G| = p_{1,1}^{a_{1,1}}$, then as in the first step of the previous proof it follows that x is a $p_{1,2}$-element and $n_1 = 2$. Again, $C_G(x) = P_{1,2} \times K$, where $P_{1,2}$ is a Sylow $p_{1,2}$-subgroup of G. As before, it follows that G is a $p_{1,2}$-Baer group and a $p_{1,1}$-Baer group. Thus $P_{1,1}P_{1,2}$ is a normal subgroup of G, see [4, Theorem A]. Also all elements of order prime to $p_{1,1}$ and $p_{1,2}$ have index prime to $p_{1,1}$ and $p_{1,2}$. So $P_{1,1}P_{1,2}$ is centralised by all $p_{1,1}^{'}, p_{1,2}^{'}$- elements of G and thus is our required A_1.$\Box$

We note that in Theorem 2 if there are more than two factors we cannot deduce that G is a Baer group if one of the factors has only one prime. In [4] we conjectured that if a group had the conjugate type vector of a nilpotent group then the group had to be nilpotent. It seems that this is harder than the situation when many possible conjugacy class sizes do not exist.

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