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ON UNIFORM EBERLEIN COMPACTA AND C-ALGEBRAS

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This paper is dedicated to the memory of Amer Bešlagić.

Abstract. We investigate a question of Y. Benyamini, M. E. Rudin and M. Wage, on the existence of universal uniform Eberlein compacta of a given weight, more exactly the related question of the existence of a universal c-algebra of a given size. We show that for any regular $\lambda > \aleph_1$ with $2^{\aleph_0} > \lambda$, there is no c-algebra of size λ universal under c-embeddings. In fact, under these circumstances, for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size λ such that every c-algebra of size λ c-embeds into one of the μ given ones.

1. Introduction.

We investigate the question of the existence of universal uniform Eberlein compacta. The question of the existence of universal uniform Eberlein compacta of a given size was asked by Y. Benyamini, M.E. Rudin and M. Wage as Problem 3 in their 1977 paper [BeRuWa]. In his recent paper [Be], M. Bell

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showed that this is equivalent to the existence of universal c-algebras under ordinary embeddings. He showed in [Be] that if $\lambda = 2^{<\lambda}$, then there is a c-algebra of size λ which is universal not just under ordinary embeddings, but also under a stronger notion of a c-embedding. We are interested in the question of the existence of such an algebra when the relevant instances of GCH fail. We show that for any regular cardinal $\lambda > \aleph_1$ with $2^{\aleph_0} > \lambda$, there can be no c-algebra of size λ into which every c-algebra of size λ c-embeds. In fact, under these circumstances, for no $\mu < 2^{\aleph_0}$ are there μ c-algebras of size λ such that every c-algebra of size λ embeds into one of the μ given ones. The niceness property of the c-algebras is not used in the above. We also show that $\clubsuit + \neg CH$ implies that there is no c-algebra of size $\aleph_1$ into which every c-algebra of size $\aleph_1$ c-embeds, in fact, there are no $\mu < 2^{\aleph_0}$ c-algebras of size $\aleph_1$ such that every c-algebra of size $\aleph_1$ c-embeds into one of them. In fact, $\clubsuit_{\rm club}$ suffices in the place of $\clubsuit$. It is a result of M. Bell from [Be] that adding $\aleph_2$ Cohen reals suffices to have a model in which there is no universal uniform Eberlein compact of weight $\aleph_1$, hence also no c-algebra of size $\aleph_1$ universal even under ordinary embeddings.

A uniform Eberlein compact, abbreviated UEC, is a topological space homeomorphic to a weakly compact subspace of a Hilbert space. A UEC X^* is said to be *universal* of weight λ iff every UEC of weight $\leq \lambda$ is a continuous image of it. The intuition suggesting that a universal object is the one into which all other objects of that kind embed is justified when one passes to the objects roughly dual to the UEC, so called c-algebras. Calling a c-algebra B^* universal iff every other c-algebra of the same size embeds into B^*, it turns out that there is a universal UEC of weight λ iff there is a universal c-algebra of size λ (M. Bell in [Be]). Note that this result does not follow immediately from the Stone duality theorem, as not every UEC is 0-dimensional.

For a history and further references on UEC, see papers [BeRuWa] and [Be]. The method of invariants used in this paper was developed in the context of linear orders by M. Kojman and S. Shelah in [KjSh 1], and they have used it in other contexts elsewhere.

2. Preliminaries.

Notation 3. *For $\alpha > \theta = \mathrm{cf}(\theta)$, we let*

$$S^\alpha_\theta \stackrel{\mathrm{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \theta\}.$$

Definition 4. *(1) A boolean algebra B is a* c-algebra *iff there is a family $\{B_n : n < \omega\}$ of subsets of B such that*

- : *(i) $n \neq m \implies B_n \cap B_m = \emptyset$,*
- : *(ii) Each B_n consists of pairwise disjoint elements,*
- : *(iii) $\bigcup_{n<\omega} B_n$ generates B,*
- : *(iv) $\bigcup_{n<\omega} B_n$ has* the nice property, *meaning that for no finite $F \subseteq \bigcup_{n<\omega} B_n$ do we have $\bigvee F = 1$.*

We say that $\langle B_n : n < \omega\rangle$ witnesses that B is a c-algebra. When discussing c-algebras, we always have in mind a fixed sequence witnessing this, although we may omit to mention it. We may refer to it as $\langle B_n(B) : n < \omega\rangle$.

(2) If B^*_l for $l \in \{0,1\}$ are c-algebras, then a 1-1 boolean homomorphism $f : B^*_0 \to B^*_1$ is a c-embedding *iff $f``B_n(B^*_0) \subseteq B_n(B^*_1)$ for all $n < \omega$. If there is such an embedding from B^*_0 to B^*_1, we write*

$$B^*_0 < B^*_1.$$

(3) A c-algebra B^* of size λ is universal *of size λ iff for any c-algebra B of size $\leq \lambda$, there is an embedding $f : B \to B^*$.*
(4) A c-algebra B^* of size λ is c-universal *of size λ iff for any c-algebra B of size $\leq \lambda$, there is a c-embedding $f : B \to B^*$.*
(5) A boolean algebra is almost c *iff it satisfies all the properties of c-algebras, except possibly for (iv) in (1) above.*

Fact 5. *(M. Bell,* [Be]*) (1) There is a universal UEC of size λ iff there is a universal c-algebra of size λ.*
(2) If $\lambda = 2^{<\lambda}$, then there is a c-universal c-algebra of size λ.

Definition 6. *Let* $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.
(1) A sequence $\langle c_\delta : \delta \in S\rangle$ *is called a* club-sequence *iff* S *is a stationary set of limit ordinals* $< \lambda$*, and for each* $\delta \in S$*, the set* c_δ *is a club subset of* δ.
(2) A club sequence $\langle c_\delta : \delta \in S\rangle$ *is called* a club-guessing sequence *iff for every club* E *of* λ *there is a stationary set of* $\delta \in S$ *such that* $c_\delta \subseteq E$.

The following fact is a result of S. Shelah from [Sh 2] [III 7.8], see also [Sh 1], and a proof is also in M. Kojman and S. Shelah's [KjSh 1].

Fact 7. *(S. Shelah,* [Sh 2]*) Suppose* $\lambda = \mathrm{cf}(\lambda) > \aleph_1$.
<u>*Then*</u> *there is a club guessing sequence* $\langle c_\delta : \delta \in S \subseteq S^\lambda_{\aleph_0}\rangle$.

Definition 8. *(1)* $\clubsuit$ *is the statement that there is a sequence*

$$\langle c_\delta : \delta \textit{ limit } < \omega_1\rangle$$

with $c_\delta \subseteq \delta = \sup(c_\delta)$ *such that for every* $A \subseteq \omega_1$ *unbounded,*

$$\{\delta : c_\delta \subseteq A\}$$

is stationary.
(2) $\clubsuit_{\mathrm{club}}$ *is the statement obtained from* $\clubsuit$ *by replacing above in "every* $A \subseteq \omega_1$ *unbounded", the word "unbounded" by "club".*

9. Non-existence of c-universal c-algebras.

Theorem 1. *Suppose that* $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ *satisfies* $2^{\aleph_0} > \lambda$.
<u>*Then*</u> *there is no c-universal c-algebra of size* λ*, moreover for no* $\mu < 2^{\aleph_0}$ *are there* μ *c-algebras of size* λ *such that every c-algebra of size* $\leq \lambda$ *c-embeds into one of them.*

Proof. Fix λ as in the hypothesis, and a club guessing sequence

$$\bar{c} = \langle c_\delta : \delta \in S \subseteq S^\lambda_{\aleph_0}\rangle$$

as guaranteed by Fact 7.

Definition 10. *(1) Suppose that* B *is a boolean algebra of size* λ*. A sequence* $\bar{B} = \langle B^\alpha : \alpha < \lambda\rangle$ *is a* filtration of B *iff*

: *(i)* $B^\alpha \subseteq B^{\alpha+1}$ *and* $B^0 = \emptyset$.
: *(ii)* $B^\delta = \bigcup_{\alpha<\delta} B^\alpha$ *for* δ *limit,*
: *(iii)* $|B^\alpha| < \lambda$,
: *(iv)* $\bigcup_{\alpha<\lambda} B^\alpha = B$.

(2) Suppose that B *is a c-algebra of size* λ *with a filtration* $\bar{B}$*, while* $\delta \in S$ *and* $b \in B \setminus B^\delta$. *We define*

$$\mathrm{Inv}_{\bar{B}}(b, c_\delta) \stackrel{\mathrm{def}}{=} \{\alpha \in c_\delta : \\ (\exists m \geq 1)(\exists y \in B_m(B) \cap B^{\min(c_\delta \setminus (\alpha+1))} \setminus B^\alpha)\,[y \geq b]\}.$$

If $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ *because* $m \geq 1$ *is such that for some* $y \geq b$ *we have* $y \in B_m(B) \cap B^{\min(c_\delta \setminus (\alpha+1))} \setminus B^\alpha$, *we say that* $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ *by virtue of* m.

Note 11. *With the notation of Definition 10:*
(1) For every $m \geq 1$*, there is at most one* $\alpha \in \mathrm{Inv}_{\bar{B}}(b, c_\delta)$ *which is there by virtue of* m *(unless* $b = 0$*).*
[Why? As elements of $B_m(B)$ *are pairwise disjoint.]*
(2) $|\{A : (\exists b \in B)(\exists \delta \in S)\mathrm{Inv}_{\bar{B}}(b, c_\delta) = A\}| \leq \lambda$.

Main Claim 12. *Suppose that for* $\delta \in S$ *we are given* $A_\delta \subseteq c_\delta$ *with* $\sup(A_\delta) = \delta$ *and* $otp(A_\delta) = \omega$.

Then there is a c-algebra B *and filtration* $\bar{B}$ *of* B *such that for all* $\delta \in S$ *there is* $b^0_\delta \in B \setminus B^\delta$ *such that* $\mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = A_\delta$.

Proof of the Main Claim. Let $f_\delta : \omega \to A_\delta$ be the increasing enumeration of A_δ. We shall use α^δ_m to stand for $f_\delta(m)$.

For $n < \omega$, let X_n be the boolean algebra generated by $\{a_\eta : \eta \in {}^{\leq n}\lambda\}$ freely except for the equations

: (1) $a_{\langle\rangle} = 1$,
: (2) $[\eta \neq \varepsilon \;\&\; \lg(\eta) = \lg(\varepsilon)] \implies a_\eta \wedge a_\varepsilon = 0$,
: (2) $\eta \trianglelefteq \varepsilon \implies a_\eta \geq a_\varepsilon$.

Let $\mathcal{B}^*$ be the product algebra $\langle X_n : n < \omega\rangle$. Let I be the ideal of eventually 0 elements of $\mathcal{B}^*$, and let $\mathcal{B} \stackrel{\mathrm{def}}{=} \mathcal{B}^*/I$.

For $\delta \in S$ let:

$$b^0_\delta \stackrel{\mathrm{def}}{=} [\langle a_{f_\delta\restriction 0}, a_{f_\delta\restriction 1}, \ldots a_{f_\delta\restriction n}, \ldots\rangle]$$

and for $m \geq 1$ let b^m_δ be the class of the function in $\mathcal{B}^*$ which is constantly equal to $a_{f_\delta \restriction m}$. Note that $b^m_\delta \geq b^0_\delta$ for all $m \geq 1$. Also note that b^0_δ are pairwise disjoint, as for $\delta_1 \neq \delta_2$ both in S, we have that $A_{\delta_1} \cap A_{\delta_2}$ is finite.

For $n < \omega$ let $B_n \stackrel{\text{def}}{=} \{b^n_\delta : \delta \in S\}$. Our algebra B is given by

$$B \stackrel{\text{def}}{=} \langle \bigcup_{n<\omega} B_n \rangle_{\mathcal{B}}.$$

It is easily seen that B is a c-algebra of size λ.

Next we define the filtration $\bar{B} = \langle B^\alpha : \alpha < \lambda \rangle$ of B by letting $B^0 = \emptyset$ and for $\alpha < \lambda$

$$B^{\alpha+1} \stackrel{\text{def}}{=} \langle \bigcup_{n<\omega} B_n \cap \{[\bar{b}] : \\ (\forall n)(\exists \eta \in {}^{\leq n}\alpha)\,[(b(n) = a_\eta \text{ or } b(n) = 0]\}\rangle_B,$$

while $B^\alpha \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} B^\beta$ for α limit.

Clearly, $\bar{B}$ is a filtration of B, and $b^0_\delta \in B \setminus B^\delta$. The proof will be finished once we prove:

Subclaim 13. *For $\delta \in S$ we have* $\mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = A_\delta$.

Proof of the Subclaim. Note that for $\delta \in S$ and $m \geq 1$ we have $b^m_\delta \in B^{\min(c_\delta \setminus (\alpha^\delta_{m-1}+1))} \setminus B^{\alpha^\delta_{m-1}}$, hence $\alpha^\delta_{m-1} \in \mathrm{Inv}_{\bar{B}}(b^0_\delta)$ by virtue of m. Now use Note 11(1). $\bigstar_{13}$ $\bigstar_{12}$

Claim 14. *Suppose that $\bar{A} = \langle A_\delta : \delta \in S \rangle$ is as in the statement of the Main Claim, and $B = B[\bar{A}]$ and $\bar{B}$ are obtained as in the Main Claim. Further suppose that $f : B \to B^*$ is a c-embedding, while $\bar{B}^* = \langle B^\alpha_* : \alpha < \lambda \rangle$ is a filtration of B^* (so $|B^*| = \lambda$).*

Then there is a club E of λ such that for every $\delta \in S$ with the property $c_\delta \subseteq E$ we have

$$A_\delta = \mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta) = \mathrm{Inv}_{\bar{B}^*}(f(b^0_\delta), c_\delta).$$

Proof of the Claim. Let

$$E \stackrel{\text{def}}{=} \{\alpha < \lambda : \ B^\alpha_* \cap f``B = f``B^\alpha\}.$$

Hence E is a club of λ.

Suppose now that $\delta \in S$ and $c_\delta \subseteq E$. Clearly

$$f(b^m_\delta) \in B_*^{\min(c_\delta \setminus (\alpha^\delta_{m-1}+1))} \setminus B_*^{\alpha^\delta_{m-1}},$$

by the definition of E, and the rest follows as in the proof of $A_\delta = \mathrm{Inv}_{\bar{B}}(b^0_\delta, c_\delta)$. $\bigstar_{14}$

Proof of the Theorem finished. Suppose that B^* is a c-algebra of size λ and $\bar{B}^*$ any filtration of B^*. As $\lambda < 2^{\aleph_0}$, we can choose for each $\delta \in S$ an unbounded $A_\delta \subseteq c_\delta$ of order type ω, such that for no $b \in B^* \setminus B^\delta_*$ do we have $\mathrm{Inv}_{\bar{B}^*}(b, c_\delta) = A_\delta$. Let $\bar{A} \stackrel{\text{def}}{=} \langle A_\delta : \delta \in S\rangle$ and $B \stackrel{\text{def}}{=} B[\bar{A}]$.

Suppose $f : B \to B^*$ is a c-embedding. Let E be a club of λ guaranteed by Claim 14, and let $\delta \in S$ be such that $c_\delta \subseteq E$. Hence $\mathrm{Inv}_{\bar{B}^*}(f(b^0_\delta), c_\delta) = A_\delta$, a contradiction.

The part of the theorem involving $\mu < 2^{\aleph_0}$ is proved similarly. $\bigstar_1$

Theorem 2. *(1) Suppose that* $\clubsuit_{\text{club}} + \neg CH$ *holds.*

Then there is no c-universal c-algebra of size $\aleph_1$. *Moreover for no* $\mu < 2^{\aleph_0}$ *are there* μ *c-algebras of size* $\aleph_1$ *such that every c-algebra of size* $\leq \aleph_1$ *c-embeds into one of them.*
(2) Suppose that either $\lambda = \mathrm{cf}(\lambda) > \aleph_1$ *satisfies* $2^{\aleph_0} > \lambda$, *or* $\lambda = \aleph_1$ *and* $\clubsuit_{\text{club}} + \neg CH$ *holds.*

Then there is no almost c-algebra of cardinality λ *which is c-universal for almost c-algebras of size* $\leq \lambda$, *or even for c-algebras of size* $\leq \lambda$. *Moreover no* $\mu < 2^{\aleph_0}$ *almost c-algebras of size* λ *taken together have that property.* $\bigstar_2$

Proof. The same proof as that of Theorem 1.

Added in proof: The author proved that similar nonuniversality results hold for other cardinals but the ones considered here, and for embeddings weaker than the one considered in this paper. These results will be reported in full elsewhere.

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