Ample Dividing

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Abstract

We construct a stable universal domain in which forking is trivial, but which has a reduct which is n-ample for all natural numbers n.

Introduction

The constructions of Hrushovski which produce new strongly minimal sets, stable $\aleph_0$-categorical structures and supersimple $\aleph_0$-categorical structures are now very familiar. In those which do not involve an infinite field, the independence relation of non-forking satisfies a property called CM-triviality. In [8], Pillay extended this notion into a hierarchy of geometric complexity for stable theories: not-1-ample (= modularity); 1-ample and not-2-ample (non-modular, CM-trivial); 2-ample etc. It is a major open problem to decide whether there are strongly minimal sets (and so on) which exhibit these various types of complexity of independence. (Recall [8] that an infinite field ∞-definable in a stable structure is n-ample for all n.) The work of Zil'ber which interprets Hrushovki's constructions in the context of complex analytic functions perhaps gives this problem additional significance.

At present, the construction of a non-CM-trivial strongly minimal set (not interpreting an infinite field) looks beyond reach, so in exploring this hierarchy we should settle for less. In [1] Baudisch and Pillay construct an ω-stable structure (of infinite rank) which is non-CM-trivial (although, as they observe, all regular types in their example are trivial). Their example is constructed as an incidence structure of points, lines and planes satisfying axioms which bear the same relation to properties of points, lines and planes in euclidean space as Lachlan's pseudoplane axioms bear to the properties of points and lines. Baudisch and Pillay therefore refer to their example as a (free) pseudospace. It is plausible that the construction of [1] could be extended to give an (infinite rank) ω-stable structure which is n-ample for $n > 2$, although the technical difficulties are already quite severe in [1].

In this paper we give a different type of construction in which there is really no additional work involved in going from 2-ampleness to n-ample for all n. Moreover, unlike in [1], n-ampleness is witnessed by points realising the same type. However, our construction achieves less than [1] in at least two important respects. Firstly, the structures we produce are not superstable. Secondly, it is not clear that the notion of stability which we are working with is full first-order stability. What is lacking (for the latter) is a model completeness result for a certain class of unary algebras (see Problem 2.5).

The structures we produce are reducts of unary algebras and in some sense this relates them to the example of [1]. The point is that the free pseudoplane is easily seen to be a reduct of the free algebra with a single unary function (cf. [7], Example 4.6.1). It would be interesting to know if the example in [1] can be seen as a reduct of a structure in which non-forking is less complicated (even trivial). The same remark applies to the (unconstrained, infinite rank) Hrushovski constructions.

In the first section we record some generalities about certain universal theories T of unary algebras with the amalgamation property. For the rest of the paper we focus on a specific T. We take a large universal domain N for T and consider a particular reduct M. When we pass to reducts of models of T the notion of substructure translates into a notion of 'nicely embedded' substructure in the reduct, denoted by $\leq$. We analyse this notion on M in Section 2. In particular we show that M has properties of universality and homogeneity for small $\leq$-embedded substructures (see Lemma 2.17). At present it is not clear how $(M; \leq)$ fits with other contexts of homogeneous models or abstract model theory (cf. [2], [3], for example), but nevertheless, we show that M supports a reasonable notion of independence with respect to which it is 'stable' (see Section 3). The main result (Theorem 3.7) is that with this notion of independence, M is n-ample for all n. We emphasise that if the theory of unary algebras we start with has a model completion, then M has a stable first-order theory which is n-ample for all n. Moreover, as it is a reduct of a structure N in which forking is trivial, no infinite group is interpretable in M. In the final section we observe that there are too many types over small sets for M to be superstable. We also note a curious connection with the predimensions used in Hrushovski's constructions, and show how to interpret a pseudospace in our examples.

1 Unary algebras

We work with a language L_1 which has only unary function symbols $f_1, f_2 \ldots$. The L_1-theory T has axioms as follows:
(i) Whenever t_1, t_2 are terms, we have an axiom $(\forall x)((t_1(t_2(x)) = x) \rightarrow (t_2(x) = x))$;
(ii) a collection of axioms of the form $(\forall x)\phi(x)$ where x is a single variable and $\phi(x)$ is quantifier-free.

So the axioms in (i) allow fixed points in a model of T, but no 'cycles' are obtained by composing the f_i.

Lemma 1.1 *The theory T has the strong amalgamation property.*

Proof. If $B_1, B_2 \models T$ have a common substructure A, then the disjoint union C of B_1, B_2 over A is an L_1-structure which is a model of T (because the axioms are all universal and involve only a single variable). $\Box$

We refer to C in the above as the *free amalgam* of B_1 and B_2 over A.

It is not true in general that T has a model companion. However, as we are dealing with a universal theory which has the amalgamation property, there is a *universal domain* N for T (cf. [6], [9]). We take this to be of large cardinality. So N has the properties:
Compactness: Any small set of q.f. formulas with parameters from N which is finitely satisfiable in N is realised in N.
Homogeneity: Small tuples with the same q.f. type lie in the same $\mathrm{Aut}(N)$-orbit.

Lemma 1.2 *The universal domain N is stable (in the sense that all q.f. formulas are stable; alternatively, the number of q.f. types over a set of size μ is $\leq \mu^{\aleph_0}$). If A, B are small substructures of N then $\mathrm{qftp}(A/B)$ does not divide over $A \cap B$.*

Proof. It is clear that if A, B are small substructures of N, then $A \cup B$ is a submodel of N, and is the free amalgam of A and B over $A \cap B$. Thus if $b \in N$ then the quantifier free type of b over A is determined by $\langle b \rangle \cap A$. The assertions follow. $\Box$

2 Reducts

Henceforth we work with a specific T. Let $s \in \mathbb{N}$, $s \geq 2$. Our language L_1 will have unary function symbols f_j^i ($i \in \mathbb{N}$, $j = 1, \ldots, s$). We write $f^i(x) = y$ as shorthand for $\bigvee_{j=1}^s f_j^i(x) = y$. Write $W_i(x, y)$ as shorthand for $(x \neq y) \wedge ((f^i(x) = y) \vee (f^i(y) = x))$.

Definition 2.1 If $i, r \in \mathbb{N}$ and A is an L_1-structure, an (i, r)-*path* in A is a sequence $a_0, \ldots, a_r$ of elements of A with $A \models W_{i+k}(a_k, a_{k+1})$ for $k \leq r-1$. It is a *nice* (i, r)-path if there is $l \leq r$ with $f^{i+k}(a_k) = a_{k+1}$ for $k < l$ and $a_k = f^{i+k}(a_{k+1})$ for $l \leq k$. We refer to a_l here as the *node* of the path. Write $A \models P^{i,r}(a, b)$ if there is an (i, r)-path in A which starts with a and ends with b.

We now define T so that the relations $P^{i,r}$ are preserved under extensions of models of T. Care has to be taken to ensure that the axioms are in the right form.

Definition 2.2 The type (ii) axioms in T are (universal closures of) the formulas $(f_j^i(x) = f_l^k(x)) \rightarrow (f_j^i(x) = x)$ whenever $(i, j) \neq (k, l)$ and a collection Θ of formulas $\theta_{i,r}$ (for $i, r \in \mathbb{N}$) expressing the following. Suppose $a_1 \in A \models \theta_{i,r}$. Suppose $a_0 = f^i(a_1)$, $a_2 = f^{i+1}(a_1), \ldots, a_r = f^{i+r-1}(a_{r-1})$ and $a_j \neq a_{j+1}$ for all $j \leq r-1$. Then there is a nice (i, r)-path in A starting with a_0 and ending with a_r.

Remarks 2.3 It is clear that $\theta_{i,r}$ can be written in the form $(\forall x)\phi(x)$ for some q.f. formula ϕ with a single variable. The following lemma shows that modulo T, the relations $P^{i,r}$ are expressible in a quantifier-free way.

Lemma 2.4 *Let $A \models T$ and suppose $a_0, \ldots, a_r$ is an (i, r)-path in A. Then there is a nice (i, r)-path in A starting at a_0 and ending at a_r. In particular, $A \models P^{i,r}(a_0, a_r) \Leftrightarrow \langle a_0, a_r \rangle \models P^{i,r}(a_0, a_r)$.*

Proof. This is by induction on r. The base case $r = 2$ follows quickly from the axioms $\theta_{i,2}$. For the inductive step, note first that we may assume $a_1, \ldots, a_r$ is a nice $(i+1, r-1)$-path, with node a_k. If $f^i(a_0) = a_1$, then $a_0, \ldots, a_r$ is a nice (i, r)-path. So suppose $f^i(a_1) = a_0$. If $k = 1$, there is no problem (we have a nice (i, r)-path with node a_0). If $k = r$ we can appeal directly to $\theta_{i,r}$ to get a nice (i, r)-path from a_0 to a_r. Finally, if $1 < k < r$ we can apply $\theta_{i,k}$

to get a nice (i,k)-path from a_0 to a_k. Adjoining $a_{k+1},\dots,a_r$ to this we get a nice (i,r)-path, as required. $\square$

By the above lemma the class of models of T does not change if we regard the W_i and $P^{i,r}$ as atomic relations. So we regard these as part of the language L_1.

Problem 2.5 Does T have a model companion?

Definition 2.6 (i) Let L denote the first-order language with 2-ary relation symbols W_i and $P^{i,r}$ (for $i,r \in \mathbb{N}$). Regard L as a subset of L_1, as outlined above.
(ii) If $A \models T$ then $A|_L$ denotes the reduct of A to L.
(iii) We denote by $\mathcal{C}_s$ (or just $\mathcal{C}$) the class of L-structures $A|_L$ where $A \models T$.

So $\mathcal{C}_s$ is a PC_Δ-class. We will find it useful to see this from a slightly different viewpoint.

Definition 2.7 We work with the class $\mathcal{G}$ of L-structures D in which the relations W_i are symmetric, irreflexive and disjoint. The interpretation of the predicates $P^{i,r}$ should be such that $D \models P^{i,r}(a,b)$ if there is an (i,r)-path in D from a to b (but not necessarily the converse). We say that an ordering $\preceq$ of such an L-structure is an *s-good ordering* (s-g.o.) if:
(i) for all $d \in D$ and $i \in \mathbb{N}$ there are at most s elements $e \in D$ with $W_i(e,d)$ and $e \preceq d$;
(ii) if $D \models P^{i,r}(a,b)$ there is an (i,r)-path $a_0 = a, a_1, \dots, a_r = b$ in D with $a_k \prec \max_{\preceq}(a,b)$ for $0 < k < r$.
Let $\mathcal{G}_s$ be the class of elements of $\mathcal{G}$ which admit an s-good ordering.

In (ii) we say that the path $a_0,\dots,a_r$ *witnesses* $D \models P^{i,r}(a,b)$ in $\preceq$.

Lemma 2.8 *Suppose $\preceq$ is an s-g.o. of $D \in \mathcal{G}_s$, and $D \models P^{i,r}(a,b)$. Then there is a path $a_0 = a, a_1, \dots, a_r = b$ witnessing this in $\preceq$ and k with $0 \le k \le r$ and $a_0 \succeq a_1 \succeq \dots \succeq a_k \preceq a_{k+1} \preceq \dots \preceq a_r$.*

Proof. This is an easy induction on r. $\square$

Definition 2.9 If $A \subseteq B \in \mathcal{G}_s$ we write $A \le B$ if there is an s-good ordering $\preceq$ of B which has A as an initial segment (i.e. if $b \in B$ then $b \preceq a \in A$ implies $b \in A$).

Lemma 2.10 *(i) The class of L-structures $A|_L$ (for $A \models T$) is precisely $\mathcal{G}_s$.*
(ii) If $C \subseteq D \in \mathcal{G}_s$ then $C \leq D$ iff there is an expansion of D to a model of T in which C becomes the domain of a substructure.

Proof. It is clear that any element of $\mathcal{C}_s$ is in $\mathcal{G}$.

First suppose that $B \models T$ and let A be a (possibly empty) substructure of B. Let $\succeq$ be the transitive closure of the 2-ary relations on B given by the graphs of the f^i_j (so $f^i_j(b) \preceq b$ for all $b \in B$). Then (by Axioms (i) of T, which forbid cycles), this is a partial order on B. Moreover no element of $B \setminus A$ comes before an element of A in this partial order. Thus $\preceq$ can be extended to a total order on B with A as an initial segment (denote this by $\preceq$ also). We claim that this is an s-good ordering of $B|_L$. Condition (i) in Definition 2.7 is clear. If $B \models P^{i,r}(a,b)$ let $a_0 = a, \ldots, a_r$ be a nice (i,r)-path in B (by Lemma 2.4). It is then easy to see that this path witnesses $B|_L \models P^{i,r}(a,b)$ in $\preceq$, as required.

Conversely suppose $D \in \mathcal{G}_s$ and $\preceq$ is an s-g.o. on D. To expand D to an L_1-structure A, we consider, for $a \in D$ the set of predecessors of a which are W_i-adjacent to it. This has size at most s, and we arbitrarily write these as $f^i_j(a)$. If there are any 'unused' f^i_j, we let $f^i_j(a) = a$. It is then easy to see that $A \models T$, (axioms Θ follows from Lemma 2.8) and $A|_L = D$. Moreover any initial segment of D under $\preceq$ is the domain of a substructure of A. $\square$

Lemma 2.11 *(i) Suppose $B \in \mathcal{G}_s$ and $\preceq$ is an s-g.o. on B with A an initial segment. Let $\preceq'$ be any other s-g.o. on A and let $\preceq''$ be defined on B by replacing $\preceq$ on A by $\preceq'$. Then $\preceq''$ is an s-g.o. on B.*
(ii) If $C \in \mathcal{G}_s$ and $A \leq B \leq C$ then $A \leq C$.
(iii) $\mathcal{G}_s$ has the strong amalgamation property for $\leq$-embeddings.

Proof. (i) Easy.
(ii) By (i).
(iii) This follows from the strong amalgamation property for T and Lemma 2.10. But we can also argue directly as follows. Suppose $A \leq B_1, B_2 \in \mathcal{G}_s$. Let C be the disjoint union of B_1 and B_2 over A. The W_i on C are given by the unions of the W_i on B_1 and B_2. For the $P^{i,r}$ on C we take the union of the $P^{i,r}$ on B_1 and B_2, and also have that $P^{i,r}(b_1, b_2)$ whenever $b_i \in B_i \setminus A$ and there is an (i,r)-path from b_1 to b_2 (and the same with the 1 and 2 reversed). It is necessary to check that if $b, b' \in B_1$ and there is an (i,r)-path from b to b' in C then there is such a path in B_1. This follows from the definition of C and Lemma 2.8 (as $A \leq B_2$). So $C \in \mathcal{G}$.

To see that $B_1 \leq C \in \mathcal{G}_s$, take an s-g.o. of B_1 with A as an initial segment. Extend the induced ordering on A to an s-g.o. of B_2, and put the elements of $B_2 \setminus A$ above those of B_1 in the ordering on C. It is easy to see that this is an s-g.o. of C. $\Box$

Definition 2.12 We refer to C in the proof of part (iii) here as the *free amalgamation* of B_1 and B_2 over A.

Definition 2.13 Suppose $B \models T$ and A is a substructure of B. Suppose C is also a model of T and $\alpha : C|_L \to A|_L$ is an isomorphism of L-structures. We define a new L_1-structure B_1 with domain B by setting $f_j^i(b) = f_j^i(b)$ if $b \in B \setminus A$ and $f_j^i(b) = \alpha(f_j^i(\alpha^{-1}(b)))$ if $b \in A$. We say that B_1 is obtained by *switching* C for A in B *via* α (omitting the latter phrase if α is the identity).

Lemma 2.14 *With the above notation $B|_L = B_1|_L$ and $B_1 \models T$.*

Proof. The first part is clear. For the second part, note that B_1 has no cycles (-it has none lying within A, or entirely outside A, and as A is a substructure of B_1, no cycle which enters A can leave it). The condition on disjointness of the graphs of the f_j^i is clear, so it remains to check the axioms Θ.

It is enough to show that if $B \models P^{i,r}(a,b)$ then there is a nice (i,r)-path in B_1 starting at a and finishing at b. If $a, b \in A$ this is clear. Take a nice (i,r)-path $a_0 = a, \ldots, a_r = b$ in B. If this has non-empty intersection with A then this intersection includes the node and is a (connected) segment $a_l, a_{l+1}, \ldots, a_m$, as A is a substructure. Now, $A \models P^{i+l,m-l}(a_l, a_m)$ so there is a nice $(i+1, m-l)$-path $a_l, a'_{l+1}, \ldots, a_m$ in αC. Inserting this into the old path in place of $a_l, \ldots, a_m$ gives a nice (i,r)-path in B_1. $\Box$

From now on, we let N be a large universal domain for T and $M = N|_L$.

Lemma 2.15 *Suppose A is a small substructure of N and N' is obtained by switching C for A in N. Then N' is a universal domain for T (and therefore isomorphic to N).*

Proof. For compactness, let $\Phi(x)$ be a small collection of q.f. formulas with parameters from the small substructure B and which is finitely satisfiable in N'. We may assume that B contains C and then that $B = C$ (- switch B for the substructure with domain B in N: the result is still N'). As $N' \models T$ and Φ is finitely satisfiable in N', there is a small $D \models T$ which contains C

as a substructure and in which there is a tuple d realising Φ. Let E be the result of switching A for C in D. By compactness of N we may assume that E is a substructure of N (containing A). If we switch C for A in N again, we obtain D as a substructure of N' (E becomes D) and the tuple d satisfies Φ, as required.

For homogeneity we observe that small partial isomorphisms of N' form a back-and-forth system. This in turn follows from the following:
Claim: If B is a small substructure of N' and also of the small model D of T there is an embedding $\alpha : D \to N'$ over B.

We know that N has this property (- a rather clumsy way of seeing this is that T has the amalgamation property and N is $|N|$-e.c. universal for T). Let $B' = C \cup B$ and $D' = D \cup C$ (the former in N', the latter the disjoint union over $B \cap C$). So $B', D' \models T$ and $C \leq B' \leq N'$. Switching A for C in B' gives $B'' \leq N$. Switching A for C in D' gives $D'' \models T$ with B'' as a substructure. So there is an embedding $\alpha : D'' \to N$ over B''. Now, D is not necessarily the domain of a substructure of D'', but when we switch C for A in D'' and N, α gives an embedding of D' into N' over B', and restricting this to D gives an embedding over B. $\square$

Definition 2.16 If A is a small subset of M we write $A \leq_1 M$ to indicate that there is an expansion of M to an L_1-structure isomorphic to N in which A is the domain of a substructure. Note that this relation is preserved by automorphisms of M.

Lemma 2.17 *(i) M has the compactness property for q.f. types, and the q.f. formulas are stable in M.*
(ii) If $A_1, A_2 \leq_1 M$ are small and $\alpha : A_1 \to A_2$ is an isomorphism (of L-structures) there is an automorphism of M which extends α.
(iii) If $A \leq_1 M$ is small, $B \in \mathcal{G}_s$ is small and $\beta : A \to B$ is an embedding such that $\beta A \leq B$, then there is an embedding $\gamma : B \to M$ such that $\gamma B \leq_1 M$ and $\gamma \circ \beta$ is the identity.
(iv) If $A \leq B \leq_1 M$ then $A \leq_1 M$.
(v) If $(A_i : i \in I)$ is a small collection of small substructures of M with $A_i \leq_1 M$ for all i, then there is a small $B \leq_1 M$ with $A_i \leq B$ for all $i \in I$.

Proof. (i) These properties are preserved under taking reducts.

(ii) Let N_i be an expansion of M to a copy of N which has A_i as a substructure. Let N_2' be the result of switching the L_1-structure $\alpha(A_1)$ for

A_2 in N_2. Let A_2' be the resulting substructure of N_2' with domain A_2. Now N_2' is isomorphic to N_1 and $\alpha : A_1 \to A_2'$ is an isomorphism of L_1-structures. Thus there is an isomorphism $N_1 \to N_2'$ which extends α. Taking the reduct to L gives an automorphism of M extending α.

(iii) We may assume that A is the domain of a substructure of N and of a substructure of an expansion of B to a model of T. The result now follows from universality of N.

(iv) Let B_1 be a model of T with domain B which has A as the domain of a substructure. We may assume B is the domain of a substructure of N. If we switch B_1 for B in N we obtain an isomorphic copy of N (by the lemma) which has A as the domain of a substructure.

(v) For $i \in I$ let N_i be an expansion of M to an isomorphic copy of N which has A_i as the domain of a substructure. There is a small subset B containing $\bigcup A_i$ which is the domain of a substructure in each of these N_i. $\square$

Corollary 2.18 *If A is a small subset of M then $A \leq M$ iff $A \leq_1 M$.*

Proof. Suppose $A \leq M$ (the other direction is obvious). Let N' be an expansion of M to a model of T in which A is the domain of a substructure. There is a small subset B containing A which is the domain of a substructure in both N' and N. So $A \leq B$ and $B \leq_1 M$. Thus $A \leq_1 M$, by Lemma 2.17(iv). $\square$

3 Independence in M

In this section we give a notion of independence $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$|$}}^*_A$ defined over small $A \leq M$ which implies non-dividing (in M) over A. Furthermore, using the terminology of [8], M is n-ample for all $n \in \mathbb{N}$ (as far as this notion is concerned). In order to interpret properly the latter in this (possibly) non-first-order context, we need a substitute for the notion of algebraic closure in M^{eq}. We take the (rather extreme) notion of 'small closure,' as defined below. If T has a model completion, then M is stable and n-ample. (The definition of n-ampleness is found in the statement of the main result here: Theorem 3.7.)

For the following, recall the definition of free amalgamation in Definition 2.12.

Definition 3.1 Suppose $A \leq B_1, B_2 \leq M$ are small. We write $B_1 \mathop{\smile\!\!\!\!\!\!|}^*{}_A B_2$ to indicate that:
(i) $B_1, B_2 \leq B_1 \cup B_2 \leq M$;
(ii) B_1, B_2 are freely amalgamated over A.

More generally, if $A \leq M$ is small and b_1, b_2 are small tuples in M we write $b_1 \mathop{\smile\!\!\!\!\!\!|}^*{}_A b_2$ to mean that there are B_1, B_2 as above containing b_1, b_2 respectively. We will not define independence over arbitrary A.

Remarks 3.2 So $B_1 \mathop{\smile\!\!\!\!\!\!|}^*{}_A B_2$ means that there is an expansion of M to a copy of N in which B_1, B_2 become the domains of substructures with intersection A.

Lemma 3.3 *In the following A, B, C, D etc. denote small $\leq$-subsets of M. The notion $\mathop{\smile\!\!\!\!\!\!|}^*$ has the following properties.*
(i) (Symmetry) Suppose $A \leq B_1, B_2 \leq M$. Then $B_1 \mathop{\smile\!\!\!\!\!\!|}^*{}_A B_2$ iff $B_2 \mathop{\smile\!\!\!\!\!\!|}^*{}_A B_1$.
(ii) (Extension) If $A \leq B, C \leq M$, there is $g \in \mathrm{Aut}(M/A)$ with $gC \mathop{\smile\!\!\!\!\!\!|}^*{}_A B$.
(iii) (Transitivity) Suppose $A \leq B \leq C \leq M$ and $D \leq M$. Then $D \mathop{\smile\!\!\!\!\!\!|}^*{}_A C$ iff $D \mathop{\smile\!\!\!\!\!\!|}^*{}_A B$ and $D \cup B \mathop{\smile\!\!\!\!\!\!|}^*{}_B C$.
(iv) (Stationarity) If $A \leq C_1, C_2 \leq M$ and $A \leq B \leq M$, and $C_i \mathop{\smile\!\!\!\!\!\!|}^*{}_A B$ and C_1, C_2 are $\mathrm{Aut}(M/A)$-conjugate, then C_1, C_2 are $\mathrm{Aut}(M/B)$-conjugate.
(v) (Local character) If $X \subseteq M$ is countable and $B \leq M$, then there are countable $A \leq B$ and $C \leq M$ with $X \subseteq C$ and $C \mathop{\smile\!\!\!\!\!\!|}^*{}_A B$.
(vi) (Non-dividing) Suppose $A \leq B, C \leq M$ and $B \mathop{\smile\!\!\!\!\!\!|}^*{}_A C$. If $(g_i : i \in I)$ is a small family of elements of $\mathrm{Aut}(M/A)$, there exists an $\mathrm{Aut}(M/A)$-conjugate $B_1 \leq M$ of B with $B_1 \mathop{\smile\!\!\!\!\!\!|}^*{}_A g_i C$ for all $i \in I$. In particular, $\mathrm{tp}_M(B_1(g_iC)) = \mathrm{tp}_M(BC)$ for all i.

Proof. (i), (ii), (iv) are clear from definitions and Lemma 2.17. Both directions of (iii) follow quite easily from the above remark. For (v), take an expansion of M to a copy of N in which B is the domain of a substructure. Let C be the substructure generated by X, and A the intersection of C and B.

Finally, (vi) follows from lemma 2.17(v) together with (ii), (iii) and (iv) here. $\square$

Remarks 3.4 If M is saturated, then (vi) shows that if $B \mathop{\smile\!\!\!\!\!\!|}^*{}_A C$ then $\mathrm{tp}(B/AC)$ does not divide over A.

Definition 3.5 Let κ be a (small) infinite cardinal. Let $G = \mathrm{Aut}(M)$. If a is a tuple of elements of M then G_a denotes the stabilizer in G of a. We fix an isomorphism type of every transitive G-space (said another way, we take representatives for G-conjugacy classes of subgroups of G and consider coset spaces for these). The *κ-small closure* of a is defined to be the union of the G_a-orbits of size less than κ on these. Denote this by $\mathrm{scl}^\kappa(a)$.

Lemma 3.6 *Suppose $A \leq B, C \leq M$ are small and $B \mathop{\smile\!\!\!\!\!\!\mid}^*{}_A C$. Then $\langle \mathrm{Aut}(M/B), \mathrm{Aut}(M/C)\rangle = \mathrm{Aut}(M/A)$. In particular, $\mathrm{scl}^\kappa(B) \cap \mathrm{scl}^\kappa(C) = \mathrm{scl}^\kappa(A)$.*

Proof. This is a standard argument. Let $H = \langle \mathrm{Aut}(M/B), \mathrm{Aut}(M/C)\rangle$. Let $g \in \mathrm{Aut}(M/A)$ and $C_1 = gC$. Let $D \leq M$ contain $C_1 \cup C$ and let B_1 have the same type over A as B and be such that $B_1 \mathop{\smile\!\!\!\!\!\!\mid}^*{}_A D$. Then $B_1 \mathop{\smile\!\!\!\!\!\!\mid}^*{}_A C$ and so by stationarity, there is an element of $\mathrm{Aut}(M/C)$ taking B to B_1. In particular, H contains $\mathrm{Aut}(M/B_1)$. But then C, C_1 are independent over A from B_1 and of the same type over A (via g). So by stationarity again, there is $h \in \mathrm{Aut}(M/B_1)$ with $h^{-1}g \in \mathrm{Aut}(M/C)$. It follows that $g \in H$, as required. $\square$

Theorem 3.7 *Suppose $s \geq 2$ and κ is any small infinite cardinal. Take $A = \{a_0, \ldots, a_n, \ldots\} \leq M$ such that $W_i(a_{i-1}, a_i)$ and $P^{i+1,(j-i)}(a_i a_j)$ (for $j > i+1$), and no other atomic relations hold on A. Then $a_i \leq M$ for each i, and for all n:*
(i) $a_n \mathop{\smile\!\!\!\!\!\!\mid}^*{}_{a_i} a_0 \ldots a_{i-1}$ for $i < n$;
(ii) $a_n \not\mathop{\smile\!\!\!\!\!\!\mid}^* a_0$, and in fact $P^{1,n}(a_0, y)$ divides over $\emptyset$;
(iii) $\mathrm{scl}^\kappa(a_0) \cap \mathrm{scl}^\kappa(a_1) = \mathrm{scl}^\kappa(\emptyset)$;
(iv) $\mathrm{scl}^\kappa(a_0 \ldots a_{i-1} a_i) \cap \mathrm{scl}^\kappa(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{scl}^\kappa(a_0 \ldots a_{i-1})$ for all i.

Proof. First note that $a_0, a_1, \ldots$ is an s-good ordering of A, so we can indeed find such points in M. Moreover, for any i, the enumeration $a_i, a_{i-1}, \ldots, a_0$ is also an s-good enumeration of the initial segment $a_0, \ldots, a_i$, so in particular $\{a_i\} \leq M$. In fact, one can now see that for any i, A is the free amalgam over a_i of $a_i, a_{i-1}, \ldots, a_0$ and $\{a_i, a_{i+1}, \ldots\} \leq A$. This gives (i).

(ii): Note that $M \models P^{1,n}(a_0, a_n)$, so $\{a_0, a_n\} \not\leq M$. Thus the first part of (ii) follows straight from the definition. However, as we are not claiming that $\mathop{\smile\!\!\!\!\!\!\mid}^*$ is really non-forking, the second part is stronger, and requires a separate argument. We do this as follows.

Let $C = \{c_i : i < \omega\}$ be the L-structure with all atomic relations empty. This is clearly in $\mathcal{G}_s$ so we may assume that $C \leq M$. Note that $c_i \leq C$, so the c_i are all $\mathrm{Aut}(M)$-conjugates of a_0 (and of course, the c_i are indiscernible over $\emptyset$). We show that no subset of $\{P^{1,n}(c_i, x) : i < \omega\}$ of size greater than s^n is realised in M. Indeed, take an s-good enumeration $\preceq$ of M with C as an initial segment. Let $d \in M$ and suppose $M \models P^{1,n}(c_i, d)$. This is witnessed by a nice $(1, n)$-path in M and (as C is an initial segment and there are no realisations of W_j in C) all vertices in this path (apart from the ends) lie between C and d in $\preceq$. It follows that there is a nice $(1, n)$-path $c_i = e_0, e_1, \ldots, e_n = d$ with $e_0 \preceq e_1 \preceq \cdots \preceq e_n$. But the number of such paths (for fixed d) is at most s^n, so the number of possible c_i reachable by such a path is at most s^n.

(iii): Suppose $e \in \mathrm{scl}^\kappa(a_0) \cap \mathrm{scl}^\kappa(a_1)$. So e lies in an $\mathrm{Aut}(M/a_0)$-orbit of size $< \kappa$ with $\omega \leq \kappa < \lambda = |M|$. There exists a sequence $(c_j : j < \kappa)$ with $c_0 = a_1$, $W_1(a_0c_j)$ for all j and $C = \{a_0\} \cup \{c_j : j < \kappa\} \leq M$ (and there are no other atomic relations on $\{a_0, c_0, \ldots\}$). Then $a_0c_j \leq M$ and the c_j are $\mathrm{Aut}(M/a_0)$-conjugate.

Claim: For some $j \neq 0$ we have that c_0, c_j are $\mathrm{Aut}(M/a_0, e)$-conjugate.

Indeed, by Lemma 2.17(i), for all $i < \kappa$ there is $g_i \in \mathrm{Aut}(M/a_0)$ with $g_ic_0 = c_i$ which permutes the elements of C. As the $\mathrm{Aut}(M/a_0)$-orbit containing e has size $< \kappa$ there exist distinct $i, k < \kappa$ with $g_ie = g_ke$. Then $g = g_i^{-1}g_k \in \mathrm{Aut}(M/a_0, e)$ and $gc_0 \neq c_0$. As the g_j permute the c_t it follows that $gc_0 = c_j$ for some $j \neq 0$, as required.

It follows that $e \in \mathrm{scl}^\kappa(c_0) \cap \mathrm{scl}^\kappa(c_j)$. But (as $s \geq 2$) any ordering of C which starts with c_0, c_j, a_0 is s-good, so $\{c_0, c_j\} \leq M$. So $c_0 \mathop{\smile\hspace{-0.9em}|}^*{}_{\emptyset} c_j$ and by Lemma 3.6, $e \in \mathrm{scl}^\kappa(\emptyset)$.

(iv): This is similar to (iii). Fix i. Let $\bar{a} = (a_0, \ldots, a_{i-1})$ and $\hat{a} = (a_{i-1}, \ldots, a_0)$. Suppose $e \in \mathrm{scl}^\kappa(\bar{a}a_i) \cap \mathrm{scl}^\kappa(\bar{a}a_{i+1})$. So e is in an $\mathrm{Aut}(M/\bar{a}a_{i+1})$-orbit of size $< \kappa$. There exist distinct $(c_j : j < \kappa)$ with $D = \{\bar{a}, a_{i+1}, c_j : j < \kappa\} \leq M$, $c_0 = a_i$, $W_i(a_{i-1}c_j)$, $W_{i+1}(c_ja_{i+1})$ and the only other instances of atomic relations holding on D being those $P^{j,r}$ forced by the (j, r)-paths. For each j, any enumeration of D starting off with $c_j, \hat{a}, a_{i+1}$ is an s-good ordering, so $c_j\hat{a}a_{i+1} \leq M$ and therefore the c_j are $\mathrm{Aut}(M/\bar{a}a_{i+1})$-conjugate. Thus (as in the proof of the claim in (iii)) we may assume c_0, c_1 are $\mathrm{Aut}(M/\bar{a}a_{i+1}e)$-conjugate. So $e \in \mathrm{scl}^\kappa(\bar{a}c_0) \cap \mathrm{scl}^\kappa(\bar{a}c_1)$. But any ordering of D starting with $\bar{a}, c_0, c_1, a_{i+1}$ is s-good (again, using $s \geq 2$) so $c_0 \mathop{\smile\hspace{-0.9em}|}^*{}_{\bar{a}} c_1$ and by Lemma 3.6 we have $e \in \mathrm{scl}^\kappa(\bar{a})$, as required. $\square$

4 Further properties of M

4.1 Non-superstability of M

In this section we show that for suitable small $A \leq M$ there are $|A|^{\aleph_0}$ $\mathrm{Aut}(M/A)$-orbits on M. So M cannot in any sense be superstable. This non-superstability can be seen within the reduct to a single W_i, and in this language there is a curious connection between s-good orderings and Hrushovski's predimensions.

Let $\mathcal{G}_s^1$ be the L-structures A where all relations $W_i, P^{i,r}$ are empty, with the possible exception of W_1 (which we will denote by W). Thus $\mathcal{G}_s^1$ consists of graphs which admit an ordering in which no vertex is adjacent to more than s of its predecessors. Note that (in contrast to $\mathcal{G}_s$), this class is closed under substructures so by general results (for example [4], Theorem 6.6.7) it is axiomatisable by universal sentences. In particular, a graph is in $\mathcal{G}_s^1$ iff all its finite subgraphs are.

Of course, this class is precisely the reducts to W_1 of the unary algebras with s functions and which have no cycles. It is not too hard to see that the e.c. models in this class of algebras are axiomatizable, so one has a model completion which is stable. Thus the corresponding reduct is stable in the full first-order sense.

Lemma 4.1 *Suppose $A \leq B \in \mathcal{G}_s^1$ and $X \subseteq B$. Then $X \cap A \leq X$.*

Proof. Take an s-g.o. of B with A as an initial segment and restrict this to X. The result is an s-g.o. of X with $X \cap A$ as an initial segment. □

It follows that each $B \in \mathcal{G}_s^1$ has a closure operation cl^B, where (for $X \subseteq B$) $\mathrm{cl}^B(X)$ is the intersection of all $A \leq B$ with $A \supseteq X$. The lemma gives that $\mathrm{cl}^B(X) \leq B$.

Lemma 4.2 *A graph B is in $\mathcal{G}_s^1$ iff every finite subgraph of B has a vertex of valency at most s (in the subgraph).*

Proof. First suppose $B \in \mathcal{G}_s^1$ and A is a finite subgraph. Take an s-g.o. on B. Then the maximal element of A with respect to this ordering is adjacent to at most s other vertices in A.

For the converse, we need only consider the case where B is finite and every subgraph has a vertex of valency at most s. We construct an s-g.o. $b_1 \preceq b_2 \preceq \cdots \preceq b_n$ of B by taking b_n any vertex of valency $\leq s$ in B and b_{n-i} any vertex of $B \setminus \{b_{n-i+1}, \ldots, b_n\}$ of valency $\leq s$ in this subgraph. □

Lemma 4.3 *If $B \in \mathcal{G}_s^1$ is finite and $X \subseteq B$ then $\mathrm{cl}^B(X)$ is the union Z of all Y with $X \subseteq Y \subseteq B$ such that the only vertices of valency $\leq s$ in the subgraph on Y lie in X.*

Proof. Note that Z also has the indicated property. We first say why $Z \leq B$. This is similar to the proof of the previous lemma. The point is that if $Z \subset A \subseteq B$ there is a vertex in $A \setminus Z$ of degree $\leq s$ in the graph on A (-consider $X \cup (A \setminus Z)$).

Next, we observe that if $X \subseteq A \leq B$ then $Z \subseteq A$. Otherwise, consider $A \cup Z$. By Lemma 4.1 $A \leq A \cup Z$. But all elements of $Z \setminus A$ which are of valency $\leq s$ in $Z \cup A$ lie in X. So no vertex of $Z \setminus A$ can be greater than A in an s-g.o. of $A \cup Z$. $\square$

We now construct continuum many graphs in $\mathcal{G}_s^1$ which are all closures of single points.

Definition 4.4 Suppose we are given the following data $\mathcal{T}$:

- a rooted, finitely branching tree T of height ω;
- a collection $(B_t : t \in T)$ of connected finite graphs, each of which is regular of valency $s+1$;
- for each $t \in T$ an edge $e_t = \{a_t, b_t\}$ of B_t;
- for each $t \in T$ and each immediate successor r of t, a vertex $v_r \in B_t$.

We write t^+ for the set of (immediate) successors of t in the tree T, and t^- for the (immediate) predecessor of t. We let $R_t = \{v_r : r \in t^+\}$, and we assume the v_r are distinct, and $a_t, b_t \notin R_t$. Furthermore, we assume that:

- R_t is a coclique in B_t
- $B_t \setminus R_t$ is connected.

We form a graph $B = B_{\mathcal{T}}$ by joining the graphs B_t together along the tree T, as follows. The vertex set of B consists of a new vertex x_0 and the disjoint union of the vertices of the B_t. The edges of B are as in the B_t, with the following exceptions:

- the edges e_t are removed;
- for each non-root vertex r in T we form new edges $\{v_r, a_r\}, \{v_r, b_r\}$;

- if t is the root of T, we form new edges $\{x_0, a_t\}, \{x_0, b_t\}$.

Lemma 4.5 *With $\mathcal{T}$ and B as above we have:*
(i) $B \in \mathcal{G}_s^1$;
(ii) $\mathrm{cl}^B(x_0) = B$;
(iii) $\mathrm{cl}^B(v_r) = \{v_r\} \cup \bigcup_{r' \geq r} B_{r'}$.

Proof. (i) Suppose for a contradiction that X is a finite subset of B on which the induced subgraph has no vertex of valency at most s. Suppose $X \cap B_t \neq \emptyset$. We show that $x_0 \in X$, which is a contradiction.

First, observe that $X \cap B_t \not\subseteq R_t$ as $s \geq 2$ and R_t is a coclique. Next, if $x \in X \cap (B_t \setminus R_t)$ then all neighbours of x lie in X as there are only $s+1$ of them. So as $B_t \setminus R_t$ is connected, we have that all vertices of $B_t \setminus R_t$ are in X, in particular, $a_t \in X$. But then it follows that $v_t \in X \cap B_{t^-}$, and we can proceed down the tree to force $x_0 \in X$.

(ii) If T' is a finite initial segment of T then x_0 is the only vertex of valency $\leq s$ in $\{x_0\} \cup \bigcup_{t \in T'} B_t$. So the statement follows from Lemma 4.3.

(iii) This is similar to (i) and (ii). $\square$

Using this it is easy to construct $2^{\aleph_0}$ non-isomorphic $\leq$-substructures of M. With only a little more effort one obtains the following.

Corollary 4.6 *For every $\nu < |M|$ there is $A \leq M$ of cardinality ν with $\nu^{\aleph_0}$ $\mathrm{Aut}(M/A)$-orbits on M.* $\square$

Remarks 4.7 Note that by the various properties of independence on M, the number of $\mathrm{Aut}(M/A)$-orbits on M is at most $|A|^{\aleph_0}$ for all small $A \subseteq M$.

We conclude by noting a curious connection between $\mathcal{G}_s^1$ and certain classes of graphs first used by Hrushovski to construct pseudoplanes.

Definition 4.8 If $k \in \mathbb{R}^{\geq 0}$ and A is a finite graph let $\delta_k(A) = k|A| - e(A)$, where $e(A)$ is the number of edges in A. For $B \subseteq A$ write $B \leq_k A$ if whenever $B \subseteq A' \subseteq A$ then $\delta_k(A') \geq \delta_k(B)$.

Remarks 4.9 Of particular interest for Hrushovski's constructions is the class of all finite graphs A with $\emptyset \leq_k A$.

Lemma 4.10 *Suppose A is a finite graph.*
(i) If $B \in \mathcal{G}_s^1$, $k \leq s/2$ and $B \leq_k A$, then $A \in \mathcal{G}_s^1$ and $B \leq A$.
(ii) If $A \in \mathcal{G}_s^1$ and $B \leq A$ then $B \leq_s A$.

Proof. (i) So we show that if $B \subset C \subseteq A$ then there is a vertex in $C \setminus B$ with valency $\leq s$ in the subgraph on C.

The sum of the valencies in C of vertices in $C \setminus B$ is $2e(C) - 2e(B)$. So as $\delta_k(C) = k|C| - e(C) \geq k|B| - e(B)$ this is at most $2k|C \setminus B|$. So the average valency in C of vertices in $C \setminus B$ is $\leq 2k \leq s$. Thus there is a vertex in $C \setminus B$ with valency $\leq s$ in C.

(ii) It is clear (by looking at how the number of edges changes as we proceed along an s-good enumeration) that $e(A) \leq e(B) + s|B \setminus A|$. So $s|B| - e(B) \leq s|A| - e(A)$, as required. $\square$

4.2 Pseudospaces in M

It is not completely clear what the precise definition of 'pseudospace' should be (the term is also not defined in [1]). Ideally, one would like to define the combinatorial notion of an 'n-pseudospace' so that a stable structure is n-ample iff it type-interprets an n-pseudospace. Of course, we have this for $n = 1$: this is Lachlan's notion of a pseudoplane. In vague terms, however, an n-pseudospace should consist of points, lines, planes, ... which satisfy various 'geometric' incidence properties.

We show how to build such a structure in M. In the example below, one could think of the loci of $a_0, \ldots, a_n$ over B as (canonical parameters for) points, lines,..., n-flats, with the various 2-types $(a_i a_j / B)$ giving incidence relations between these.

Proposition 4.11 *Let M be the structure constructed in the previous sections, with $s = 2$.*
(i) There are points $A = \{a_i, b_{i+1}, c_i, d_{i+1} : i < \omega\}$ with $A \leq M$ and only the following atomic relations (and the instances of the $P^{(i,r)}$ they imply) on A (see Figure 1): for $i \geq 1$
$W_i(a_{i-1}, a_i)$, $W_i(b_i, b_{i+1})$, $W_{i+1}(c_{i-1}, c_i)$, $W_i(a_{i-1}, c_{i-1})$, $W_i(b_i, a_i)$, $W_i(d_i, b_i)$, $W_{i+1}(d_i, c_i)$.

If $B = \{b_{i+1}, c_i, d_{i+1} : i < \omega\}$, then $B \leq A$.

With this notation, we have, for all $i < \omega$:
(ii) $a_i \notin \mathrm{acl}(B a_0, \ldots, a_{i-1}, a_{i+1}, \ldots)$;
(iii) the locus of (a_i, a_{i+1}) over B is a pseudoplane.

Proof. (i) Using Figure 1 to identify the instances of the relations $P^{i,r}$, one

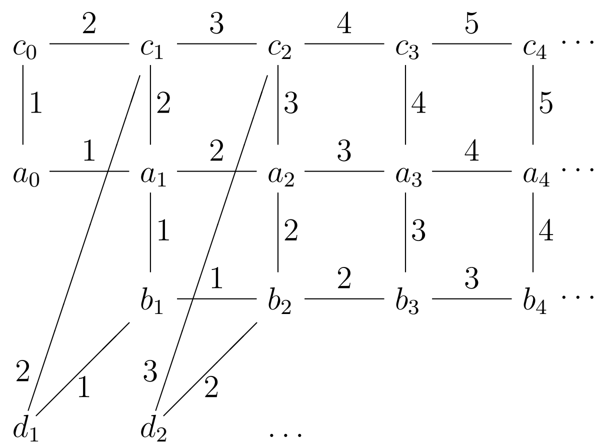

Figure 1: The pseudospace

checks that

$$d_1, d_2, \ldots, b_1, b_2, \ldots, c_0, c_1, c_2, \ldots, a_0, a_1, a_2, \ldots$$

is a 2-good ordering of A.

(ii) Let $i < \omega$. Consider the structure $E = A \cup \{e_j : j < \omega\}$ where the quantifier free type of e_j over $A \setminus \{a_i\}$ is the same as that of a_i, and there are no other basic relations on E other than what is implied by this. Then $Ba_0, a_1, \ldots, e_0, e_1, e_2, \ldots$ is a 2-g.o. of E with A as an initial segment, so we may assume that the e_j are in M. We may interchange a_i with any of the e_j and still have a 2-g.o. of E. Thus a_i, e_j lie in the same $\mathrm{Aut}(M/Ba_0, \ldots, a_{i-1}, a_{i+1}, \ldots)$-orbit.

(iii) Suppose a_i' is a translate of a_i over Ba_{i+1}. We need to check that $a_{i+1} \in \mathrm{acl}(a_i a_i')$. Indeed, suppose $a_{i+1} = a_{i+1}^1, \ldots, a_{i+1}^r$ are translates over $Ba_i a_i'$. If $r \geq 3$, then the graph with edge set W_{i+1} on the points b_{i+1}, a_i, a_i', $a_{i+1}^1, \ldots, a_{i+1}^r$ has all vertices being of valency at least 3, which contradicts the existence of a 2-g.o. on M. Thus $r \leq 2$.

Similarly, suppose a_{i+1}' is a translate of a_{i+1} over Ba_i, and $a_i = a_i^1, \ldots, a_i^r$ are translates over $Ba_{i+1}a_{i+1}'$. Again, if $r \geq 3$ then the graph with edge set W_{i+1} on the points $c_i, a_{i+1}, a_{i+1}', a_i^1, \ldots, a_i^r$ has all vertices of valency ≥ 3, which is again a contradiction. $\Box$

Remarks 4.12 Conditions (ii) and (iii) are probably weaker than n-ampleness. In the example we also have that:
(iv) $a_0, \ldots, a_{i-1} \mathop{\smile\!\!\!\!\!\!\mid}^*{}_{Ba_i} a_{i+1} \ldots$.

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