Model-theoretic constructions via amalgamation and reducts

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Introduction and Motivation

In these talks I will discuss Hrushovski's construction from 1988 of a new family of strongly minimal theories [**7**]. I will do this from the more recent viewpoint of the papers [**3, 4**].

Recall that a complete first-order theory with infinite models is strongly minimal if in any of its models, every parameter-definable subset of the model is finite or cofinite. Classical examples are theories of vector spaces and algebraically closed fields; also the degenerate example of the theory of infinite 'pure' sets where the only structure comes from equality. Algebraic closure in a strongly minimal structure satisfies the exchange condition, so gives rise to notions of dimension and independence (corresponding to linear dimension/ independence and transcendence degree/ algebraic independence in the two classical examples).

At one point, there was a conjecture of Zilber which, roughly speaking, said that any strongly minimal structure should be very closely related to one of the classical examples: more precisely, the dimension should either be (locally) modular, or an algebraically closed field should be interpretable in the structure. The point of Hrushovski's construction was to refute this. (The underlying ideology behind the conjecture was that structures which, in some sense are the most basic from a model-theoretic viewpoint, should already exist elsewhere in mathematics. The extent to which this ideology still remains intact is an interesting question: for some discussion, one can read the introduction to [**4**] and some of the references given there.)

Hrushovski's construction is a ingenious variation of the Fraïssé amalgamation class construction. It is best understood in two parts. First, one defines a certain free amalgamation class; then one restricts to a subclass, where a harder amalgamation result is needed: this second stage is usually referred to as the 'collapse' of the first part. The first stage is interesting in itself and will be the main focus of these talks. Its outcome is a structure which is not strongly minimal, but which is stable (in fact, ω-stable of infinite Morley rank). In this context, the correct notion of independence that should be considered is rather more subtle than algebraic closure: it is Shelah's notion of non-forking. But even at this level

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of generality, the uncollapsed Hrushovski structures give us something new: stable structures which do not interpret an infinite field, and where independence is not one-based (the appropriate generalisation of not being locally modular).

Independence in the Hrushovski constructions (both the strongly minimal and uncollapsed versions) has a property called CM-triviality: this is weaker than being one-based but it prevents the interpretation of an infinite field (see [**8**], for example). It is a major open problem to decide whether there are strongly minimal structures which are not CM-trivial and which do not interpret an infinite field. At present, this looks beyond reach: the feeling is that they do exist, but to construct them would require much more than a straightforward variation on Hrushovski's original idea. What we shall discuss in these talks is a method (from [**3**]) of constructing stable (but sadly not ω-stable) structures which do not interpret an infinite field and which are not CM-trivial.

One curious feature of the method in [**3**] is that the structures are obtained as reducts of other stable structures which are one-based (and also trivial). Thus by 'forgetting' some of the structure we make independence behave in a more complicated way. Another slightly surprising aspect to this is that the uncollapsed Hrushovski structures can themselves be seen as reducts of such structures: so the method is, in a rather roundabout way, a generalisation of the uncollapsed Hrushovski construction.

(Note for those who know what the words mean: I am not going to say anything about fusions or where the predimension is anything other than the simplest form: 'size of set minus some linear function of number of atomic relations'.)

The plan of the lectures is to:

- give an example of an uncollapsed Hrushovski construction and show how it can be seen as a reduct of a 'simpler' structure;
- look at the model theory of these examples (axiomatization, stability, nature of independence);
- use similar ideas to produce non-CM-trivial stable structures.

The first of these is elementary in that it needs almost no knowledge of model theory. For the second, I will not assume that you know about stability and forking (though I will need to assume some basic model theory): in fact the examples can be used to illustrate the basic definitions in the subject. The same is true of the third section, though at this stage things will become more technical: in principle things can be deduced more-or-less from the definitions, but a certain amount of faith in the meaningfulness of these definitions is required.

Prerequisites: Basic model theory including definable sets; types; algebraic closure; imaginaries; some understanding of using back-and-forth to prove completeness and understand types.

Notational conventions: Usually don't distinguish notationally between a structure and its domain (and sometimes this will be a bit confusing...)

1. Two examples of Fraïssé - Hrushovski constructions

The construction method being used here is a generalisation of Fraïssé's idea of an amalgamation class. The aim is to build a structure with an understandable model-theory out of a class of finite structures and embeddings between them. Rather than giving a general version of the construction I will give two examples.

Example 1.1 (The class of 2-out digraphs). First, some terminology. We work in the language which has a binary relation symbol $V(x, y)$, pronounced 'y is a descendant of x.' Let T' be the first-order theory whose models are the simple, loopless directed graphs (digraphs) in which all vertices have at most two descendants.

The models of T' are the *2-out digraphs*. If B is one of these and $X \subseteq B$ then we write $\mathrm{cl}'_B(X)$ for the closure of X in B under taking descendants and write $X \sqsubseteq B$ if $X = \mathrm{cl}'_B(X)$ (i.e. X is descendant-closed in B). Note that this closure is *disintegrated*:

$$\mathrm{cl}'_B(X) = \bigcup_{x \in X} \mathrm{cl}'_B(\{x\})$$

.

Let $\mathcal{D}$ be the class of 2-out digraphs. The following is just a matter of checking the definitions:

Lemma 1.2. *For $D, E \in \mathcal{D}$ we have:*

(i) *If $C \sqsubseteq D$ and $X \subseteq D$ then $C \cap X \sqsubseteq X$.*
(ii) *If $C \sqsubseteq D \sqsubseteq E$ then $C \sqsubseteq E$.*
(iii) *(Full Amalgamation) Suppose $D, E \in \mathcal{D}$ and C is a sub-digraph of both D and E and $C \sqsubseteq E$. Let F be the disjoint union of D and E over C (with no other directed edges except those in D and E). Then $F \in \mathcal{D}$ and $D \sqsubseteq F$.* □

We refer to F in the above as the *free amalgam* of D and E over C.

Using this we have:

Proposition 1.3. *There exists a countably infinite digraph N satisfying the following properties:*
(D1): *N is the union of a chain of finite sub-digraphs*
$C_1 \sqsubseteq C_2 \sqsubseteq C_3 \sqsubseteq \cdots$ all in $\mathcal{D}$.
(D2): *If $C \sqsubseteq N$ is finite and $C \sqsubseteq D \in \mathcal{D}$ is finite, then there is an embedding $f : D \to N$ which is the identity on C and has $f(D) \sqsubseteq N$.*

Moreover, N is uniquely determined up to isomorphism by these two properties and is $\sqsubseteq$-homogeneous (i.e. any isomorphism between finite closed subdigraphs extends to an automorphism of N). □

The proof of this is a variant on the classic argument of Fraïssé, and we just give a sketch. We build N as the union of a chain of 2-out digraphs as in (D1) in such a way that (D2) holds. Suppose we have constructed C_i and $C \sqsubseteq C_i$, and $C \sqsubseteq D \in \mathcal{D}$, as in (D2). Using amalgamation, we can take C_{i+1} to be the amalgam of C_i and D over C and $f : D \to C_{i+1}$

the obvious embedding. As there are only countably many isomorphism types of pairs $C \sqsubseteq D$ which can arise here, we can arrange that all tasks in (D2) are solved at some point during the construction.

The proof of the 'Moreover' part is by a back-and-forth argument. If N' also satisfies the two properties, then the set of isomorphisms $f : A \to B$ between finite $A \sqsubseteq N$ and $B \sqsubseteq N'$ is a back-and-forth system. For example, given $f : A \to B$ and $c \in N$ we can find a finite $A_1 \sqsubseteq N$ containing A and c (using D1 in N). We can then extend f to an isomorphism $g : A_1 \to B_1$ for some $B_1 \sqsubseteq N'$, using D2 in N'.

We refer to N given by the above as the *Fraïssé limit* of the amalgamation class $(\mathcal{D}, \sqsubseteq)$.

Example 1.4 (Uncollapsed Hrushovski construction). Here we work with a language which has a binary relation symbol $W(x, y)$, pronounced 'x and y are adjacent.' We work with undirected, loopless graphs. If A is a finite graph we let $e(A)$ be the number of edges in A (i.e. the number of unordered adjacent pairs of vertices) and define

$$\delta(A) = 2|A| - e(A).$$

Let T^δ be the theory of graphs in which $\delta(A) \geq 0$ for all finite subgraphs A, and let $\mathcal{C}$ be the class of all models of T^δ. If $A \subseteq B \in \mathcal{C}$ are finite we write $A \leq B$ and say that A is *self-sufficient* in B if $\delta(A) \leq \delta(B')$ whenever $A \subseteq B' \subseteq B$. Note that we can express the condition that $A \in \mathcal{C}$ by saying $\emptyset \leq A$.

The key property of the function δ is:

Submodularity: If B, C are finite subgraphs of a graph D then

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

Moreover there is equality here iff B, C are freely amalgamated over $B \cap C$ (i.e. there no adjacencies between $B \setminus C$ and $C \setminus B$).

Lemma 1.5. *We have:*

(i) *If $A \leq B$ and $X \subseteq B$ then $A \cap X \leq X$.*
(ii) *If $A \leq B \leq C$ then $A \leq C$.*
(iii) *(Full amalgamation) Suppose $A, B \in \mathcal{C}$ and C is a subgraph of A and B and $C \leq B$. Let D be the disjoint union (i.e. free amalgam) of A and B over C. Then $D \in \mathcal{C}$ and $A \leq D$.*

Proof: All are quick proofs using submodularity. For example, for (i), take $A \cap X \subseteq Y \subseteq X$. Then $\delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(A \cap Y)$. Rearranging, we get $0 \leq \delta(A \cup Y) - \delta(A) \leq \delta(Y) - \delta(A \cap X)$, as $A \leq B$. $\square$

Note (i) and (ii) here imply that if $A, B \leq C$ then $A \cap B \leq C$. Thus for every $X \subseteq B$ there is a smallest self-sufficient subset of B which contains X. Denote this by $\mathrm{cl}_B(X)$: the self-sufficient closure of X in B. For infinite graphs $A \subseteq B$ we can define $A \leq B$ to mean $A \cap X \leq X$ for all finite $X \subseteq B$. The lemma also holds, and (i) shows that is consistent with the definition in the finite case, so we can talk about self-sufficient closure in this more general context. Note that if X is a finite subset of $C \in \mathcal{C}$ then $\mathrm{cl}_C(X)$ is finite (we look at finite subsets $X \subseteq Y \subseteq C$ and choose one where $\delta(Y)$ is as small as possible: then $Y \leq C$ and $\mathrm{cl}_C(X) \subseteq Y$).

As in the previous example, we get:

PROPOSITION 1.6. *There is a countably infinite graph M satisfying the following properties:*

(C1): *M is the union of a chain of finite subgraphs*
$B_1 \leq B_2 \leq B_3 \leq \cdots$ all in $\mathcal{C}$.
(C2): *If $B \leq M$ is finite and $B \leq C \in \mathcal{C}$ is finite, then there is an embedding $f : C \to M$ which is the identity on B and has $f(C) \leq M$.*

Moreover, M is uniquely determined up to isomorphism by these two properties and is $\leq$-homogeneous (i.e. any isomorphism between finite self-sufficient subgraphs extends to an automorphism of M). □

The two examples give amalgamation classes $(\mathcal{D}, \sqsubseteq)$ and $(\mathcal{C}, \leq)$ from which we construct respectively, as Fraïssé limits, a countable digraph N and a countable graph M.

THEOREM 1.7. *If we forget the direction on the edges in N, the resulting graph is isomorphic to M.*

Thus M is a *reduct* of N (in the sense that there is a bijection $M \to N$ which sends $\emptyset$-definable sets to $\emptyset$-definable sets in all powers).

DEFINITION 1.8. Suppose A is a graph. A *$\mathcal{D}$-orientation* of A is a directed graph $A^+ \in \mathcal{D}$ with the same vertex set as A and such that if we forget the direction on the edges, we obtain A. We say that $A_1, A_2 \in \mathcal{D}$ are *equivalent* if they have the same vertex set and the same graph-reduct (i.e. they are $\mathcal{D}$-orientations of the same graph).

The Theorem is a fairly straightforward corollary of the following two lemmas:

LEMMA 1.9. *(1) Suppose B is a finite graph. Then $B \in \mathcal{C}$ iff B has a $\mathcal{D}$-orientation.*
(2) If $B \in \mathcal{C}$ and $A \subseteq B$, then $A \leq B$ iff there is a $\mathcal{D}$-orientation of B in which A is closed.

LEMMA 1.10. *(1) If $C \sqsubseteq D \in \mathcal{D}$ and we replace the digraph structure on C by an equivalent structure $C' \in \mathcal{D}$, then the resulting digraph D' is still in $\mathcal{D}$.*

(2) If $A \leq B \in \mathcal{C}$ then any $\mathcal{D}$-orientation of A extends to a $\mathcal{D}$-orientation of B. □

Part (1) of the second of these is straightforward: just observe that any vertex in D' has at most two directed edges coming from it. Part (2) then follows from (2) of the first lemma and (1).

The proof of the first lemma uses a classical theorem of combinatorics, usually referred to as Hall's Marriage Theorem:

THEOREM 1.11. *Suppose S is a set, I is a finite set and for each $i \in I$ we have a finite subset $X_i \subseteq S$. Suppose further that for every $J \subseteq I$ we have*

$$|\bigcup_{j \in J} X_j| \geq |J|.$$

Then there exists a sequence of distinct elements $(x_i : i \in I)$ such that $x_i \in X_i$. □

Proof of 1.9: (1) First suppose B has a $\mathcal{D}$-orientation B^+, and let $Y \subseteq B$. This also has an orientation (induced by that on B). Now count directed edges in the orientation (which is of course the same as $e(Y)$): each vertex has at most two directed edges coming from it, so the number of them is $\leq 2|Y|$. Thus $\delta(Y) \geq 0$, so $B \in \mathcal{C}$.

Conversely, suppose $B \in \mathcal{C}$. Let I denote the set of edges in B. For each $i = \{a, b\}$ we wish to choose a direction of the edge. This is equivalent to selecting one of a, b as the initial vertex of the directed edge; and for the resulting digraph to be in $\mathcal{D}$ we should select the same vertex no more than twice as i varies. Let B' be another copy of the vertex set B and $S = B \cup B'$. For $i = \{a, b\} \in I$ let $X_i = \{a, b, a', b'\}$ (where a', b' denote the copies of a, b in B'). Now suppose $J \subseteq I$ and let $Y = \bigcup J$ (the vertices in these edges). Then $\delta(Y) \geq 0$, so

$$|J| \leq e(Y) \leq 2|Y| = |\bigcup_{j \in J} X_j|.$$

Thus the condition in the Marriage Theorem is satisfied, and we can find distinct $(x_i : i \in I)$ with $x_i \in X_i$. To orient the edge $i = \{a, b\}$ we look at x_i: if $x_i = a$ or a' then we make a the initial vertex. As the x_i are distinct, any vertex is chosen no more than twice in this way.

(2) The proof is just a modification of the above. □

Now we prove Theorem 1.7. Let N^- denote the graph-reduct of the digraph N. Property (D1) in N and Lemma 1.9 imply that N^- has property (C1). To prove the Theorem it suffices (by proposition 1.6) to show that (C2) also holds in N^-.

Suppose $B \leq N^-$ and $B \leq C \in \mathcal{C}$. Let D^+ be the $\sqsubseteq$-closure of B in N. So D^+ is a finite 2-out digraph, and B inherits a $\mathcal{D}$-orientation B^+ from it. By Lemma 1.10, this extends to a $\mathcal{D}$ orientation of C in which B^+ is $\sqsubseteq$-closed: call the resulting digraph C^+. By Full Amalgamation for $\mathcal{D}$ (Lemma 1.2(iii)) the free amalgam E^+ of D^+ and C^+ over B^+ is in $\mathcal{D}$ and $D^+ \sqsubseteq E^+$. So by (D2) there is a digraph embedding $f^+ : E^+ \to N$ which is the identity on D^+ and with $f^+(E^+) \sqsubseteq N$. Now pass to the graph-reduct (indicated in the notation by dropping the $^+$ superscript): E is the free amalgam of C and D over B and as $B \leq D$ we have $C \leq E$. We have a graph embedding $f : E \to N^-$ which is the identity on B and $f(C) \leq f(E) \leq N^-$, as required.

Remarks 1.12. (1) One can see that this is a general argument. One has a class $(\mathcal{D}, \sqsubseteq)$ of relational structures and distinguished embeddings which (amongst other things) satisfies full amalgamation. By taking reducts one obtains another class $(\mathcal{C}, \leq)$ of structures and distinguished embeddings where the definition of $A \leq B$ can be taken as saying that there is an expansion of B to a structure $B^+ \in \mathcal{D}$ so that $A \sqsubseteq B^+$. The extra condition which one needs to make this work is the analogue of Lemma 1.10.

(2) In [**7**] Hrushovski works with structures with a ternary relation R and a predimension $\delta(A) = |A| - |R[A]|$. This leads to a structure of Morley rank ω which can be collapsed to a structure of rank 1. Our graph M has rank $\omega.2$ and can be collapsed to a structure of rank 2. I've chosen to work with graphs and digraphs here as they're rather more intuitive than ternary structures and directed ternary structures. But the same sort of thing can be done for the original Hrushovski example: see [**4**].

2. Model theory of the constructions

2.1. The theory of N.

2.1.1. *Axiomatization and understanding types.* Recall that T' is the theory of 2-out digraphs in the language L' which has a 2-ary relation symbol $V(x,y)$ for 'there is a directed edge from x to y.' We want to axiomatise $T_N = Th(N)$ in this language, where N is the Fraïssé limit in Proposition 1.3.

Suppose $X \sqsubseteq A \in \mathcal{D}$ are finite. Let $\bar{x}$ and $\bar{y}$ be tuples of variables with the variables in $\bar{x}$ corresponding to the elements of X and the variables in $\bar{y}$ corresponding to the elements of $A \setminus X$. Let $\Delta_X(\bar{x})$ and $\Delta_{X,A}(\bar{x},\bar{y})$ denote the basic diagrams of X and A repectively and $\phi_{X,A}$ the L'-sentence:

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \to \Delta_{X,A}(\bar{x},\bar{y}) \wedge \text{`cl}'(\bar{x}\bar{y}) = \text{cl}'(\bar{x}) \cup \bar{y}\text{'})$$

The condition '$\text{cl}'(\bar{x}\bar{y}) = \text{cl}'(\bar{x}) \cup \bar{y}$' is expressed in a first-order way by saying that any descendant of a variable in $\bar{y}$ is one of the variables in $\overline{xy}$.

Let T_1' consist of (the deductive closure of) T' together with all these $\phi_{X,A}$. Thus a model N_1 of T' is a model of T_1' iff for all finite subsets X of N_1 and $X \sqsubseteq A \in \mathcal{D}$ with A finite, there is an embedding over X of A into N_1 whose image A_1 has closure $\text{cl}'_{N_1}(X) \cup A_1$. Note that by compactness we have also have the following. Suppose N_1 is an ω-saturated model of T_1', $X \sqsubseteq N_1$ is the closure of a finite set, and $X \sqsubseteq A \in \mathcal{D}$ where A is the closure of a finite set. Then there exists an embedding over X of A into N_1 with closed image.

LEMMA 2.1. *The theory T_1' is consistent and complete. In fact, $Th(N) = T_1'$. Moreover, n-tuples $\bar{a}, \bar{b}$ in models N_1, N_2 of T_1' have the same types iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\text{cl}'_{N_1}(\bar{a})$ and $\text{cl}'_{N_2}(\bar{b})$.*

Proof: First we show that $N \models \phi_{X,A}$. This follows from *full* amalgamation in $(\mathcal{D}, \sqsubseteq)$. More precisely, given $X \subseteq N$ finite and $X \sqsubseteq A \in \mathcal{D}$ (also finite) let $Y = \text{cl}'_N(X)$. Then $Y \sqsubseteq N$ is finite. Let B be the free amalgam of Y and A over X. So $Y \sqsubseteq B$ is the closure of A in B. By (D2) in Proposition 1.3, there is a closed copy of B over Y in N: this witnesses the truth of $\phi_{X,A}$ for this particular X.

If the types of $\bar{a}$ and $\bar{b}$ are the same, then clearly we have an isomorphism between their closures. For the rest, it is enough to show that if N_1, N_2 are ω-saturated models of T_1', then the set of isomorphisms between closures of finite subsets of N_1 and N_2 is a back-and-forth system (cf. [**12**], Chapitre 5.b or [**13**], Section 5.2). But this follows at once from the remarks immediately preceding the lemma. □

2.1.2. *Stability and non-forking.* Suppose N_1 is a highly saturated model of T_N. If $B \sqsubseteq N_1$ is small and $\bar{a}$ is a tuple of elements of N_1 then by the above $\text{tp}(\bar{a}/B)$ is determined by $\text{cl}'(\bar{a}B)$, which is the free amalgam of $A = \text{cl}'(\bar{a})$ and B over their intersection. Let's count the number of possible types here. Note that A is countable, so the number of possibilities for $A \cap B$ is $|B|^{\aleph_0}$. The number of possibilities for the isomorphism type of A over $A \cap B$ is at most $2^{\aleph_0}$, so the number of n-types over B is at most $max(2^{\aleph_0}, |B|^{\aleph_0})$.

Definition 2.2. Suppose λ is an infinite cardinal. A complete theory T in a countable language is λ-*stable* if for every model M of T and $B \subseteq M$ with $|B| \leq \lambda$, the number of 1-types over B is at most λ. We say that T is *stable* if it is λ-stable for some λ.

Thus our calculation shows that T_N is λ-stable whenever $\lambda = \lambda^{\aleph_0}$, so T_N is stable.

A stable theory has saturated models of arbitrarily large cardinality. It's convenient to have the convention that we fix a large saturated model (the 'monster model') and all models of T we want to discuss are regarded as elementary submodels of this and all parameter sets are small subsets of it.

Definition 2.3. Suppose T is a complete stable theory in a countable language L and $\psi(\bar{x}, \bar{b})$ a consistent L-formula (with parameters $\bar{b}$). Say that $\psi(\bar{x}, \bar{b})$ *divides over* C if there exist a natural number k and distinct realisations $(\bar{b}_i : i < \omega)$ of $\mathrm{tp}(\bar{b}/C)$ such that no k of the formulas $\{\psi(\bar{x}, \bar{b}_i) : i < \omega\}$ are consistent.

If $C \subseteq B$ we say that $\mathrm{tp}(\bar{a}/B)$ divides over C if there is some $\psi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{a}/B)$ which divides over C.

If $\mathrm{tp}(\bar{a}/B)$ does not divide over C we write $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$: pronounced $\bar{a}$ is independent from B over C.

If you've never seen this before it's a bit hard to understand. Roughly the idea is that independence means that '$\bar{a}$ is as free as possible from B over C.' It's far from obvious that $\mathop{\smile\!\!\!\!\!\!|}$ has any nice properties, but it does. Suppose we make the notation more symmetric by writing $A \mathop{\smile\!\!\!\!\!\!|}_C B$ to mean that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ for every finite tuple $\bar{a}$ from A. Then, assuming (harmlessly) that $A \supseteq C$, we have the symmetry property:

$$A \mathop{\smile\!\!\!\!\!\!|}_C B \Leftrightarrow B \mathop{\smile\!\!\!\!\!\!|}_C A.$$

Another property of independence is that if C is a model and $\bar{a}$ and $\bar{a}'$ have the same type over M and each is independent from B over C, then $\bar{a}$ and $\bar{a}'$ have the same type over B. In fact if we include imaginary elements, the same is true for any algebraically closed C.

(We should also remark that there is a notion called *forking* which is in general weaker than dividing, but which coincides with it in stable (and simple) theories. It would be more usual to pronounce $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_C B$ as '$\mathrm{tp}(\bar{a}/B)$ does not fork over C.')

To illistrate the definitions, we prove the following.

Lemma 2.4. *With the above notation, and working in a large saturated model N_1 of T_N, if B is closed in N_1, then* $\mathrm{tp}(\bar{a}/B)$ *does not divide over* $C = \mathrm{cl}'(\bar{a}) \cap B$.

Proof. Suppose $(B_i : i < \omega)$ is a sequence of translates of B over C (more formally, we regard these as being enumerated as tuples which have the same type over C as some fixed enumeration of B). Note that $B_i \sqsubseteq N_1$. Let X be the union of these and let Y be the free amalgam of X and $\mathrm{cl}'(\bar{a})$ over C. Then $X \sqsubseteq N_1$ and $X \sqsubseteq Y$ so we may assume that $Y \sqsubseteq N_1$. Let $\bar{a}_1$ be the copy of $\bar{a}$ inside Y. Then $\mathrm{cl}'(\bar{a}_1) \cap B_i = C$ and $\mathrm{cl}'(\bar{a}_1)$ and B_i are freely amalgamated over C. Thus $\mathrm{tp}(\bar{a}_1 B_i) = \mathrm{tp}(\bar{a}B)$ for all i. This proves the lemma. $\square$

Corollary 2.5. *If A, B, C are subsets of a model of T_N then*

$$A \underset{C}{\downarrow} B \Leftrightarrow \mathrm{cl}'(AC) \cap \mathrm{cl}'(BC) = \mathrm{cl}'(C).$$

Proof. The direction $\Leftarrow$ follows from the Lemma. For the other direction note that $\mathrm{cl}'(AC)$ is equal to the algebraic closure of AC in N_1 and we have in general:

Lemma 2.6. *For any stable theory, $A \downarrow_C B$ implies that* $\mathrm{acl}(AC) \cap \mathrm{acl}(BC) = \mathrm{acl}(C)$. □

(Here algebraic closure can be taken to include imaginary elements: we do not need this here, but we shall use it later.) □

2.2. Model theory of M.

2.2.1. *Axiomatization and types.* We now do the same sort of thing for the structure M constructed in Example 1.4. So T^δ is the theory of graphs satisfying $\delta \geq 0$ on all finite subgraphs, in the language L^δ with a binary relation symbol $W(x, y)$. The axiomatization of $T_M = Th(M)$ is similar to the axiomatization of $Th(N)$, but with one extra complication.

For a natural number m and finite $Y \subseteq Z \in \mathcal{C}$ we write $Y \leq^m Z$ to mean that $\delta(Y) \leq \delta(Z')$ whenever $Y \subseteq Z' \subseteq Z$ and $|Z' \setminus Y| \leq m$. It's easy to check that Lemma 1.5 holds if we replace $\leq$ throughout by $\leq^m$. In particular there is a closure cl^m associated to $\leq^m$.

Suppose $X \leq A \in \mathcal{C}$ are finite. As before, let $\bar{x}$ and $\bar{y}$ be tuples of variables with the variables in $\bar{x}$ corresponding to the elements of X and the variables in $\bar{y}$ corresponding to the elements of $A \setminus X$. Let $\Delta_X(\bar{x})$ and $\Delta_{X,A}(\bar{x}, \bar{y})$ denote the basic diagrams of X and A repectively and for each natural number m let $\sigma^m_{X,A}$ be the L^δ-sentence:

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \to \Delta_{X,A}(\bar{x}, \bar{y}) \wedge \text{`}\mathrm{cl}^m(\bar{x}\bar{y}) = \mathrm{cl}^m(\bar{x}) \cup \bar{y}\text{'})$$

It takes a little bit of thought to see that the condition '$\mathrm{cl}^m(\bar{x}\bar{y}) = \mathrm{cl}^m(\bar{x}) \cup \bar{y}$' can be expressed in a first-order way.

Let T^δ_1 be (the deductive closure of) T^δ and these $\sigma^m_{X,A}$. So a model M_1 of T^δ is a model of T^δ_1 iff for all finite subsets X of M_1 and $X \leq A \in \mathcal{C}$ (with A finite) and all m there is an embedding over X of A into M_1 whose image A_1 has m-closure $\mathrm{cl}^m_{M_1}(X) \cup A_1$. By compactness if M_1 is ω-saturated and $X \leq M_1$ is finite and $X \leq A \in C$ is finite, then there is an embedding of A over X into M_1 whose image A_1 is closed in M_1. The proof of the following is then as in Lemma 2.1.

Lemma 2.7. *The theory T^δ_1 is consistent and complete. In fact, $Th(M) = T^\delta_1$. Moreover, n-tuples $\bar{a}, \bar{b}$ in models M_1, M_2 of T^δ_1 have the same type iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\mathrm{cl}_{M_1}(\bar{a})$ and $\mathrm{cl}_{M_2}(\bar{b})$.* □

2.2.2. *Stability and independence.* We now want to count types in a highly saturated model M_1 of T_M. Suppose $B \leq M_1$ and $\bar{a}$ is a tuple in M_1. There is a finite $C \leq B$ with $\delta(\mathrm{cl}(\bar{a}C)) - \delta(C)$ as small as possible (– this is an integer ≥ 0) and $\mathrm{cl}(\bar{a}C) \cap B = C$. For such a C we claim:

Claim: $\mathrm{cl}(\bar{a}C) \cup B \leq M_1$ and is the free amalgam of $\mathrm{cl}(\bar{a}C)$ and B over C.

Proof of Claim: Let $A = \mathrm{cl}(\bar{a}C)$. It suffices to prove the claim when B is finite (– by considering finite closed subsets of the original B). By definition of δ if A, B are not freely amalgamated over C then $\delta(\mathrm{cl}(\bar{a}B)) \leq \delta(A \cup B) < \delta(A) + \delta(B) - \delta(C)$, which, after rearranging the inequality, contradicts the choice of C. We have a similar contradiction if $\delta(\mathrm{cl}(\bar{a}B)) < \delta(A \cup B)$, thus $A \cup B \leq M_1$. □

By the claim we see that $\mathrm{tp}(\bar{a}/B)$ is determined by C and the isomorphism type of $\mathrm{cl}(\bar{a}C)$. So the number of 1-types over B is at most $max(\aleph_0, |B|)$. Thus T_M is λ-stable for all infinite λ.

Theorem 2.8. *If $A, B, C \subseteq M_1 \models T_M$ then $A \mathop{\smile\!\!\!\!\!\mid}_C B$ iff*

- $\mathrm{cl}(AC) \cap \mathrm{cl}(BC) = \mathrm{cl}(C)$
- $\mathrm{cl}(AC)$ *and* $\mathrm{cl}(BC)$ *are freely amalgamated over* $\mathrm{cl}(C)$
- $\mathrm{cl}(ABC) = \mathrm{cl}(AC) \cup \mathrm{cl}(BC)$.

Sketch of Proof. Assuming the 3 conditions hold, the proof that $A \mathop{\smile\!\!\!\!\!\mid}_C B$ is very similar to the proof of Lemma 2.4. To simplify the notation we can assume that A, B are closed and have intersection C and we can assume that M_1 is highly saturated. We show that $\mathrm{tp}(A/B)$ does not divide over C. Suppose $(B_i : i < \omega)$ is a sequence of translates of B over C. Let X be the closure of the union of these and let Y be the free amalgam of X and A over C. As $B_i \leq X$ we have that A and B_i are freely amalgamated over C and $A \cup B_i \leq Y$. We may assume that $Y \leq M_1$. If A' denotes the copy of A in Y then $\mathrm{tp}(A'B_i) = \mathrm{tp}(AB)$ for each i.

For the converse, we can use Lemma 2.6 and the fact that algebraic closure in M_1 is self-sufficient closure to obtain the first bullet point if $A \mathop{\smile\!\!\!\!\!\mid}_C B$. Moreover, we can assume as before that A, B are closed and have intersection C. To simplify the argument, assume also that A, B are finite. Let $(B_i : i < \omega)$ be a sequence of translates of B over A which are freely amalgamated over C and such that the union of any subcollection of them is self-sufficient in M_1. Suppose for a contradiction that A, B are not freely amalgamated over C. Then the same is true of A and B_i and there is $s > 0$ such that $\delta(A \cup B_i) = \delta(A) + \delta(B_i) - \delta(C) - s$ for all i. Then one computes that

$$\delta(A \cup \bigcup_{i=1}^{r} B_i) \leq \delta(\bigcup_{i=1}^{r} B_i) + \delta(A) - C - rs.$$

If r is large enough, this contradicts $\bigcup_{i=1}^{r} B_i \leq M_i$. The third bullet point is similar. □

2.3. Triviality, one-basedness and CM-triviality. Each of these three properties limits the 'complexity' of independence in a stable theory. To the non-specialist, they will look rather technical, but we give the definitions and illustrate them in the two examples.

Definition 2.9. Suppose T is a complete, stable theory. We say that T is *trivial* if the following holds over any parameters. Whenever a, b, c are tuples of elements from a model of T which are pairwise independent, then $a \mathop{\smile\!\!\!\!\!\mid} b, c$ (over the parameters).

Example 2.10. We show that T_N is trivial. Indeed, let N_1 be a (large, saturated) model of T_N, suppose a, b, c are tuples from N_1 and suppose we are working over some closed parameter set D. Let A, B, C denote $\mathrm{cl}'(aD)$ etc. To say that a, b, c are pairwise independent over D means that A, B, C have pairwise intersection D. Because closure is disintegrated in N_1, we have $\mathrm{cl}'(B \cup C) = B \cup C$, so $A \mathop{\smile\!\!\!\!\!\mid}_D B \cup C$.

Example 2.11. We show that T_M is non-trivial. There exists $\{a, b, c, d\} \leq M$ such that each of a, b, c is adjacent to d, but there are no other adjacencies. A simple calculation shows that $d \in \mathrm{cl}(a, b, c)$. Then (using the description of independence), a, b, c are pairwise independent (over $\emptyset$) but $a \not\mathop{\smile\!\!\!\!\!\mid} b, c$ because $\{a, b, c\} \not\leq M$.

Definition 2.12. A complete, stable theory T is *one-based* if whenever A, B are algebraically closed sets in a model of T we have $A \mathop{\smile\!\!\!\!\!\mid}_{A \cap B} B$.

Caveat: Here 'algebraic closure' means in the sense of T^{eq}: it includes imaginary elements.

It can be shown that our examples T_M and T_N have *weak elimination of imaginaries*, which means we can view algebraic closure as being taken in the models, without having to consider imaginary elements. That said, we can see that T_N is one-based, but T_M is not (for the latter, take $A = \{a\}$ and $B = \{b, c\}$ as in the example above).

Definition 2.13. A complete, stable theory T is *CM-trivial* if whenever A, B, D are algebraically closed sets in a model of T and A, B are independent over their intersection, then $A \cap D$ and $B \cap D$ are independent over their intersection.

Again, algebraic closure here is in the sense of T^{eq}. It should be clear that one-basedness implies CM-trivialty. It is fairly easy to check, given the description of independence in Theorem 2.8 and assuming weak elimination of imaginaries, that T_M is CM-trivial.

There are various equivalent ways of defining CM-triviality given in [**7**]. Another reformulation was given by Pillay [**10**] where one-basedness and CM-triviality are given as two steps in a hierarchy of complexity of independence in a stable theory. Not being one-based is called 1-ampleness in this hierarchy; not being CM-trivial is called 2-ampleness. We will not give the general definition of n-ampleness, but note the following:

Definition 2.14. A complete, stable theory T is 2-*ample* if, possibly after adding parameters, there exist tuples a, b, c in some model of T such that:

(i) $a \not\mathop{\smile\!\!\!\!\!\mid} c$
(ii) $a \mathop{\smile\!\!\!\!\!\mid}_b c$
(iii) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$ and $\mathrm{acl}(ab) \cap \mathrm{acl}(ac) = \mathrm{acl}(a)$.

Again, algebraic closure should be in the sense of T^{eq} here. We omit the proof that 2-ampleness is equivalent to non-CM-triviality for a stable theory (see [**10**]).

Remarks 2.15. (1) We have shown that T_N is a trivial, one-based stable theory with a reduct T_M which is neither one-based nor trivial. The fact that triviality is not preserved under reducts answers a question from [**5**].

(2) It has been known for a long time that one-basedness is not preserved under reducts: probably this is due to Hodges (see [**6**] or [**2**]). Essentially Hodges' example is what we have done, but with 1-out digraphs.

(3) In [**1**], Baudisch and Pillay construct an ω-stable, trivial, non-CM-trivial structure.

(4) Pillay has recently shown [**11**] that the theory of a finitely generated non-abelian free group (– known to be stable by work of Sela) is not CM-trivial. He has conjectured that there is a simple group of finite Morley rank which is 2-ample but not 3-ample (in particular, it would be a counterexample to the Algebraicity Conjecture on groups of finite Morley rank). Algebraically closed fields are n-ample for all n.

(5) A general presentation of the Hrushovski constructions and their stability can be found in [**14**].

3. Avoiding CM-triviality

I now want to use the technique of taking a reduct of the Fraïssé limit of a suitable amalgamation class to produce a stable structure which is non-trivial, non-CM-trivial. At the beginning the idea of the construction is a little bit difficult to motivate: so I will try to do this only after we have gone some distance into it.

To avoid unnecessary decoration of symbols, I will re-use symbols such as $\mathcal{D}, \mathcal{C}, \leq, \sqsubseteq, \ldots$ from previous sections, giving them new meaning, but the same context.

3.1. Directed structures. We work with a first-order language in a signature consisting of two binary relation symbols $V_R(x, y), V_B(x, y)$, pronounced 'y is a red descendant of x' and 'y is a blue descendant of x' respectively. The class $\mathcal{D}_0$ consists of structures in which these relations are disjoint, and each of V_R, V_B gives a digraph in which all vertices have at most 2 descendants. If $C \subseteq D \in \mathcal{D}_0$ we write $C \sqsubseteq D$ if C is closed under taking red and blue descendants in D. Again we write $\mathrm{cl}'_D(X)$ for the descendant closure of X in D.

Write $R(x, y)$ iff $V_R(x, y) \vee V_R(y, x)$. Write $B(x, y)$ iff $V_B(x, y) \vee V_B(y, x)$.

We will again consider undirected reducts, but we will also retain information about the existence of certain paths (of length 2, of the form RB) between pairs of vertices when we pass to the reduct. In the directed graphs the existence of such a path between two vertices is not in general preserved between closed substructures. So we impose extra conditions on the class in order to guarantee this.

Definition 3.1. Let θ be the closed formula which says:

> if x, y, z are such that $V_R(z, x) \wedge V_B(z, y)$, then there is w such that $V_R(x, w) \wedge V_B(w, y)$ or $V_B(y, w) \wedge V_R(w, x)$ or $V_R(x, w) \wedge V_B(y, w)$.

Let $\mathcal{D}$ be the class of structures in $\mathcal{D}_0$ which satisfy this.

Clearly $\mathcal{D}$ is an elementarty class: i.e. the class of models of some theory T_D.

If $a, b, c \in D \in \mathcal{D}_0$ satisfy $V_R(a,b) \wedge V_B(b,c)$ or $V_B(c,b) \wedge V_R(b,a)$ or $V_R(a,b) \wedge V_B(c,b)$, then we refer to a, b, c as a *nice RB-path* from a to c. The axiom θ says that if there is an RB-path from a to c, then there is a nice RB-path from a to c. Note that such a path lies in the $\sqsubseteq$-closure of a, c thus we have:

Lemma 3.2. *If $C \sqsubseteq D \in \mathcal{D}$, then $C \in \mathcal{D}$.*

As in Section 1 we have:

Lemma 3.3. *For $D, E \in \mathcal{D}$ we have:*

(i) *If $C \sqsubseteq D$ and $X \subseteq D$ then $C \cap X \sqsubseteq X$.*
(ii) *If $C \sqsubseteq D \sqsubseteq E$ then $C \sqsubseteq E$.*
(iii) *(Full Amalgamation) Suppose $D, E \in \mathcal{D}$ and C is a sub-digraph of both D and E and $C \sqsubseteq E$. Let F be the disjoint union of D and E over C (with no other directed edges except those in D and E). Then $F \in \mathcal{D}$ and $D \sqsubseteq F$.* □

We refer to F in the above as the *free amalgam* of D and E over C. It requires a little bit of thought to see that F satisfies θ.

Using this we have:

Proposition 3.4. *There exists a countably infinite $N \in \mathcal{D}$ satisfying the following properties:*
(D1): *N is the union of a chain of finite sub-digraphs*
$C_1 \sqsubseteq C_2 \sqsubseteq C_3 \sqsubseteq \cdots$ all in $\mathcal{D}$.
(D2): *If $C \sqsubseteq N$ is finite and $C \sqsubseteq D \in \mathcal{D}$ where D is finite, then there is an embedding $f : D \to N$ which is the identity on C and satisfies $f(D) \sqsubseteq N$.*

Moreover, N is uniquely determined up to isomorphism by these two properties and is $\sqsubseteq$-homogeneous (i.e. any isomorphism between finite closed substructures extends to an automorphism of N). □

3.2. The reduct.

Consider the following definable predicate $P(x, y)$. For $a, b \in C \in \mathcal{D}$ we write

$$C \models P(a,b) \Leftrightarrow C \models \exists z(R(a,z) \wedge B(z,b)).$$

The point of the axiom θ is to ensure:

Lemma 3.5. *If $a, b \in B \sqsubseteq C \in \mathcal{D}$ then*

$$B \models P(a,b) \Leftrightarrow C \models P(a,b).$$

Proof. We need to consider the possibility that c witnessing $C \models P(a,b)$ lies in $C \setminus B$. As $B \sqsubseteq C$ we must have $C \models V_R(c,a) \wedge V_B(c,b)$. But then θ guarantees that $B \models P(a,b)$. □

Definition 3.6. We let $\mathcal{C}$ be the class of structures in the language $\{R, B, P\}$ which arise as a reduct of a structure in $\mathcal{D}$. For $D \in \mathcal{D}$ we might write D^- for the reduct in $\mathcal{C}$ and we say that the directed structure D is an *orientation* of D^-. Structures in $\mathcal{D}$ with the same domain are *equivalent* if their reducts are the same. If $A \subseteq B \in \mathcal{C}$ we write $A \leq B$ to mean that there is an orientation of B in which the subset A is a closed subset.

Warning: The class $\mathcal{C}$ is not closed under substructures. For example, take $C = \{a, b, c\} \in \mathcal{D}$ with $C \models V_R(a, c) \wedge V_B(b, c)$. Then $C \models P(a, b)$, so $C^- \models P(a, b)$. But clearly $\{a, b\}$ itself is not the reduct of a structure in $\mathcal{D}$. Nevertheless, by the definition, if $A \leq B \in \mathcal{C}$ then $A \in \mathcal{C}$.

Lemma 3.7. *(1) (Switching) Suppose $A \sqsubseteq C \in \mathcal{D}$ and C_1 is obtained from C by replacing the substructure on A by an equivalent structure A_1. Then $C_1 \in \mathcal{D}$ and C_1 is equivalent to C.*

(2) If $A \leq C \leq D \in \mathcal{C}$ then $A \leq D$.

(3) (Free amalgamation) If $A \leq C, D \in \mathcal{C}$ then the free amalgam F of C and D over A is in $\mathcal{C}$ and $C, D \leq F$.

We need to be more precise about what we mean by F here. The domain is the disjoint union of C and D over A. The R and B relations are just those in C and D, but in addition to the P-relations in C, D we also have $F \models P(b, c)$ when $b \in D \setminus A$ and $c \in C \setminus A$ and there is $a \in A$ with $R(a, b) \wedge B(a, c)$ (and similarly with the roles of D, C interchanged).

Proof of Lemma. (1) It is easy to see that $\mathcal{C}_1 \in \mathcal{D}_0$, so we check that $C_1 \models \theta$. To do this we show that if there is a nice RB-path a, b, c from a to c in C, then there is a nice RB-path a, b', c from a to c in C_1. If $a, b, c \in C \setminus A$ there is nothing to do. If $a, c \in A$ then there is an RB-path from a to c in A_1, so there is a nice RB-path in A_1 as $A_1 \in \mathcal{D}$. We are left to consider the case where one of a, c is in A, say $a \in A$ and $c \notin a$. If $b \notin A$ there is again nothing to prove as none of the directions in the path gets changed when going from C to C_1. So we are left with the case where $b \in A$. As $A \sqsubseteq C$ we have $C \models V_B(c, b)$ and the same is true in C_1. So whatever direction is on the R-edge between a and b, the RB-path a, b, c is nice.

The statement about equivalence is clear: the relations R and B are the same in both C and C_1, and the relation P is determined by these.

(2) There is an orientation D^+ of D in which C is closed and an orientation C^+ of C in which A is closed. Using (1), we can replace the oriented structure on C in D^+ by C^+ and obtain an orientation of D in which A is closed.

(3) Take an orientation D^+ of D in which A is closed. By (1) and the fact that $A \leq C$, the induced orientation A^+ on A extends to an orientation C^+ of C. It is then easy to see that the free amalgam of D^+ and C^+ over A^+ is an orientation of F. □

Notice that (2) and (3) follow from (1) and the corresponding properties in $(\mathcal{D}, \sqsubseteq)$.

Remarks 3.8. One difference from the previous case is that there is no closure operation associated with $\leq$: it can happen that $A_1, A_2 \leq C \in \mathcal{C}$ and $A_1 \cap A_2 \not\leq C$. For example, suppose C has points a, b_1, b_2, c and relations $R(a, b_i)$, $B(c, b_i)$, $P(a, c)$ (for $i = 1, 2$). One can check easily (by producing orientations) that $C \in \mathcal{C}$ and $\{a, b_1, c\}, \{a, b_2, c\} \leq C$. On the other hand $\{a, c\} \not\leq C$, as $P(a, c)$.

3.3. Model theory of N.

As in the monochrome case in Section 2 we can axiomatise $T_N = Th(N)$ by taking T_D together with axioms of the form

$$\forall \bar{x} \exists \bar{y} (\Delta_X(\bar{x}) \to \Delta_{X,A}(\bar{x},\bar{y}) \wedge \text{`cl}'(\bar{x}\bar{y}) = \text{cl}'(\bar{x}) \cup \bar{y}\text{'})$$

where $A \in \mathcal{D}$ is finite, $X \sqsubseteq A$, $\Delta_X(\bar{x})$ denotes the basic diagram of X and $\Delta_{X,A}(\bar{x},\bar{y})$ denotes the basic diagram of A, where the variables $\bar{y}$ represent the elements of $A \setminus X$. The condition '$\text{cl}'(\bar{x}\bar{y}) = \text{cl}'(\bar{x}) \cup \bar{y}$' is expressed in a first-order way by saying that any $V_{R/B}$-descendent of a variable in $\bar{y}$ is one of the variables in $\overline{xy}$.

Note that if X is the closure of a finite set inside some ω-saturated model N' of T_N, and $X \sqsubseteq A \in \mathcal{D}$, where A is also the closure of a finite set, then, by compactness, there exists an embedding over X of A into N' with closed image. One can then argue exactly as before to obtain:

LEMMA 3.9. *(i) The theory T_N is consistent and complete. Moreover, n-tuples $\bar{a}, \bar{b}$ in models N_1, N_2 of T_N have the same type iff the map $\bar{a} \mapsto \bar{b}$ extends to an isomorphism between $\text{cl}_{N_1}(\bar{a})$ and $\text{cl}_{N_2}(\bar{b})$.*
(ii) The theory T_N is stable and if A, D, C are subsets of a model N' of T_N, then $A \mathop{\smile\!\!\!\!\!\!|}_C D \Leftrightarrow \text{cl}'_{N'}(AC) \cap \text{cl}'_{N'}(DC) = \text{cl}'_{N'}(C)$. Moreover, T_N is one-based and trivial. □

3.4. Model theory of the reduct. We now consider the undirected reduct N^- of N. We do not have an axiomatisation of $T_C = Th(N^-)$. The difficulty is that although N^- is the Fraïssé limit of the finite structures in $(\mathcal{C}, \leq)$, this class does not have full amalgamation and so the limit is more difficult to axiomatize. Nevertheless, as the reduct of a stable theory, it is stable; and it is the reduct of a complete, recursively axiomatized theory, it is decidable. We investigate T_C by working more closely with T_N.

Henceforth we work with a large saturated model $\hat{N}$ of T_N. We let $\hat{M}$ denote its RBP-reduct. So $\hat{M}$ is a saturated model of T_C. In the following, 'small' means of cardinality less than $|\hat{N}|$.

PROPOSITION 3.10. *Suppose $A \sqsubseteq \hat{N}$ is small and $A_1 \in \mathcal{D}$ is equivalent to A. Let $\hat{N}_1$ be the structure obtained by replacing A by A_1 in $\hat{N}$. Then $\hat{N}_1$ is a saturated model of T_N.*

Proof. By Lemma 3.7 (i) we have $\hat{N}_1 \in \mathcal{D}$. So it will be be enough to show that $\hat{N}_1$ satisfies the following 'genericity' condition. Suppose $B \sqsubseteq \hat{N}_1$ and $B \sqsubseteq D \in \mathcal{D}$ is small. Then there is an embedding $\delta : D \to \hat{N}_1$ which is the identity on B, and which satisfies $\delta(D) \sqsubseteq \hat{N}_1$. Indeed, if this condition holds, then $\hat{N}$ and $\hat{N}_1$ are back-and-forth equivalent, so $\hat{N}_1 \models T_N$, and saturation is then clear from the description of types in Lemma 3.9.

As A_1 and B are closed in $\hat{N}_1$ we have $B_1 = A_1 \cup B \sqsubseteq \hat{N}_1$. Let D_1 be the free amalgam over B of B_1 and D. So $A_1 \sqsubseteq B_1 \sqsubseteq D_1$ and $D \sqsubseteq D_1 \in \mathcal{D}$. If we replace A_1 by the equivalent structure A in B_1 we obtain $A \sqsubseteq B_2 \sqsubseteq \hat{N}$. Doing the same thing in D_1 we obtain $D_2 \in \mathcal{D}$ (by Lemma 3.7) with $A \sqsubseteq B_2 \sqsubseteq D_2$. By saturation of $\hat{N}$ (i.e. the above genericity property), there is an embedding $\alpha : D_2 \to \hat{N}$ which is the identity on B_2 and which has closed image in $\hat{N}$. Now, D is not necessarily the domain of a closed substructure of D_2, but if we replace the structure on A by A_1 in both D_2 and $\hat{N}$, the map α gives us an embedding $D_1 \to \hat{N}_1$ (- same map, different structures!) with closed image and which is the identity on B_1. If we restrict this to $D \sqsubseteq D_1$, we get the required embedding δ. □

Corollary 3.11. *(i) If $A \subseteq \hat{M}$ is small, then $A \leq \hat{M}$ iff there is an orientation of $\hat{M}$ which is a saturated model of T_N in which A is closed.*
(ii) If $A \leq \hat{M}$ is small and $\beta : A \to B$ is an embedding of A into some small $B \in \mathcal{C}$ with $\beta(A) \leq B$, then there exists an embedding $\gamma : B \to \hat{M}$ with $\gamma \circ \beta$ the identity on A and $\gamma(B) \leq \hat{M}$.
(iii) If $A_1, A_2 \leq \hat{M}$ are small and $\alpha : A_1 \to A_2$ is an isomorphism, then α can be extended to an automorphism of $\hat{M}$.

Proof. (i) Suppose $A \leq \hat{M}$ is small. Let Q be an orientation of $\hat{M}$ in which A is closed. There is a small subset B containing A which is closed in both Q and $\hat{N}$. Let B_1 denote the structure on B in Q. So $A \sqsubseteq B_1$. Replace the structure on B in $\hat{N}$ by the equivalent structure B_1. By Proposition 3.10 the result is still a saturated model $\hat{N}_1$ of T_N. So we have $A \sqsubseteq B_1 \sqsubseteq \hat{N}_1$ and $\hat{N}_1$ is an orientation of $\hat{M}$ which is saturated and in which A is closed.

(ii) This follows from (iii) and the fact that any small $B \in \mathcal{C}$ can be $\leq$-embedded in $\hat{M}$.

(iii) By Proposition 3.10 and (i), there exist orientations $\hat{N}_1$, $\hat{N}_2$ of $\hat{M}$ which are saturated models of T_N with A_1, A_2 (respectively) closed subsets and in which α gives an isomorphism of the oriented structures on A_1, A_2. By Lemma 3.9 (i) this is a partial elementary map, so by uniqueness of saturated models, it extends to an isomorphism between $\hat{N}_1$ and $\hat{N}_2$. Passing back to the reduct, we obtain an automorphism of $\hat{M}$ which extends α. □

We do not have a full characterization of forking in $\hat{M}$. However, the following is useful.

Lemma 3.12. *Suppose A, B, C are small subsets of $\hat{M}$ with $A \cap B = C \leq \hat{M}$; $A, B \leq A \cup B \leq \hat{M}$ and $A \cup B$ the free amalgam over C of A and B. Then $A \mathop{\smile\!\!\!\!\!\mid}_C B$.*

Proof. This is similar to the proof of Lemma 2.8: we show that $\mathrm{tp}_{\hat{M}}(A/B)$ does not divide over C. Let $(B_i : i < \omega)$ be a sequence of translates over C of $B = B_0$. So in particular $B_i \leq \hat{M}$. First, we show that there is a small $D \leq \hat{M}$ with $B_i \leq D$ for all $i < \omega$. To see this, note that for each i there is an orientation $\hat{N}_i$ of $\hat{M}$ in which C and B_i are closed. As the closure of a small set is small in any orientation, there is a small subset D which contains all the B_i and which is closed in $\hat{N}$ and all the $\hat{N}_i$. It follows that $B_i \leq D \leq \hat{M}$ for all i.

Let F be the free amalgam over C of D with a copy over C of A (call it A_1). By Corollary 3.11(ii), we may assume that $F \leq \hat{M}$. As $A_1 \leq F \leq \hat{M}$ we have that A and A_1 have the same type over C. Now we claim that $A_1, B_i \leq A_1 \cup B_i \leq F$ for each i. Indeed, there is an orientation D' of D in which C, B_i are closed. Extend the orientation C' on C to an orientation A_1' of A_1 (using Lemma 3.7). The free amalgam (in $\mathcal{D}$) of A_1' and D' over C' is an orientation of F in which A_1, B_i and $A_1 \cup B_i$ are closed. The establishes the claim and also shows that $A_1 \cup B_i$ is the free amalgam over C of A_1 and B_i. Thus $\mathrm{tp}_{\hat{M}}(B_i A_1) = \mathrm{tp}_{\hat{M}}(BA)$ for all i, by Corollary 3.11(iii). □

Again, note that the proofs of these are formal, given the switching property and propertites of $(\mathcal{D}, \sqsubseteq)$.

4. 2-ampleness of the reduct

We continue to use the notation of the previous subsection. Let $A = \{a, b, c\} \leq \hat{M}$ be such that $\hat{M} \models R(a, b) \wedge B(b, c) \wedge P(a, c)$. We have:

THEOREM 4.1. *The structure $\hat{M}$ is non-trivial and 2-ample. In fact:*

(i) $a \mathop{\smile\!\!\!\!\!\mid}_b c$;
(ii) $a \not\mathop{\smile\!\!\!\!\!\mid} c$ *and in fact $P(a, y)$ divides over $\emptyset$;*
(iii) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$;
(iv) $\mathrm{acl}(ab) \cap \mathrm{acl}(ac) = \mathrm{acl}(a)$.

Proof. Non-triviality can be seen just by using the relation R (as in non-triviality of the Hrushovski structure in Section 1). To prove 2-ampleness, we verify (i)-(iv).

(i) We can orient A so that $V_R(a, b) \wedge V_B(c, a)$. Thus $\{b\}, \{a, b\}, \{c, b\} \leq \hat{M}$, and so (i) follows from Lemma 3.12.

(ii) As $P(a, y) \in \mathrm{tp}_{\hat{M}}(c/a)$, it is enough to prove that $P(a, y)$ divides over $\emptyset$. As in (i) but using a different orientation we have that $\{c\} \leq \hat{M}$. Let $C = \{c_i : i < \omega\} \leq \hat{M}$ have all atomic relations P, B, R empty on C. Thus $c_i \leq \hat{M}$ for each i, and these are indiscernible and of the same type as c over $\emptyset$. We show that there is no $a' \in \hat{M}$ with $\hat{M} \models \bigwedge_{i=1}^{5} P(a', c_i)$. Suppose there is such an a'. There is an orientation of $\hat{M}$ in which C is $\sqsubseteq$-closed. In this there is a RB-path from a' to each c_i and as C is $\sqsubseteq$-closed (and there are no atomic relations on C) these paths can be taken to be good RB-paths of the form $V_R(a, b_i) \wedge V_B(b_i, c_i)$ for some b_i. The b_i need not be distinct here, but $c_1, \ldots, c_5$ are distinct: this is impossible as each vertex has at most 2 red descendants and 2 blue descendants in the orientation.

(iii) Suppose $e \in \mathrm{acl}(a) \cap \mathrm{acl}(b)$. There is a sequence $\{b_j : j < \omega\}$ of distinct elements of $\hat{M}$ with $b = b_0$, $\hat{M} \models R(a, b_j)$ for each j, and no other atomic relations holding on $B = \{a, b_0, b_1 \ldots\} \leq \hat{M}$. Then $ab_j \leq \hat{M}$ and the b_j are all of the same type over a (by automorphisms). The same is true of any pair of the b_j. As $e \in \mathrm{acl}(a)$ it follows that b_0, b_1 have the same type over ae.

Thus $e \in \mathrm{acl}(a) \cap \mathrm{acl}(b_1)$, so $e \in \mathrm{acl}(b_0) \cap \mathrm{acl}(b_1)$. Now, Lemma 3.12 shows that $b_0 \mathop{\smile\!\!\!\!\!\mid} b_1$, which implies $\mathrm{acl}(b_0) \cap \mathrm{acl}(b_1) = \mathrm{acl}(\emptyset)$.

(iv) This is similar to (iii). Take $e \in \mathrm{acl}(ab) \cap \mathrm{acl}(ac)$. There exist distinct $(b_j : j < \omega)$ with $b = b_0$ and $D = \{a, c, b_j : j < \omega\} \leq \hat{M}$ with $R(a, b_j) \wedge B(b_j, c)$ for all j being the only atomic relations on D (apart from $P(a, c)$). To see this, note that the orientation with

$$V_R(a, b_0) \wedge V_B(c, b_0) \wedge \bigwedge_{j>0} V_R(b_j, a) \wedge V_B(b_j, c)$$

is in $\mathcal{D}$. By replacing b_0 by any of the other b_j, we see that $ab_jc \leq D \leq \hat{M}$. In particular, the b_j are of the same type over ac. As $e \in \mathrm{acl}(ac)$ we can therefore assume that b_0, b_1 are of the same type over ace, so $e \in \mathrm{acl}(ab_0) \cap \mathrm{acl}(ab_1)$. By choosing a different orientation we can see $ab_0 \leq ab_0b_1 \leq ab_0b_1c \leq \hat{M}$ (– take an orientation of D where b_0, b_1 are red descendants of a and blue descendants of c). Thus by Lemma 3.12 again, $b_0 \downarrow_a b_1$ and so $e \in \mathrm{acl}(a)$. □

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