CONSTRUCTING CONTINUUM MANY COUNTABLE, PRIMITIVE, UNBALANCED DIGRAPHS

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Abstract. We construct continuum many non-isomorphic countable digraphs which are highly arc-transitive, have finite out-valency and infinite in-valency, and whose automorphism groups are primitive.
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1. Introduction and Notation

By an *unbalanced* digraph we mean a directed graph in which vertices have finite out-valency and infinite in-valency. A digraph is *primitive* if its automorphism group is primitive on the set of vertices, that is, the only equivalence relations on the vertex set which are preserved by the automorphism group are the trivial ones.

Primitive unbalanced digraphs were first constructed in [4], answering a question of Peter M. Neumann from [6]. The construction there gives countably many countable examples. Neumann subsequently asked (private communication) whether there are continuum many countable, primitive unbalanced digraphs. The main result of this paper (Theorem 2.14) is a positive answer to this question. We show that there are continuum many pairwise non-isomorphic highly arc transitive directed graphs in which each vertex has finite out-valency and infinite in-valency, and whose automorphism group is primitive on vertices and transitive on directed edges.

The digraphs which we construct are highly arc transitive: the automorphism groups are transitive on the set of n-arcs, for all finite n (and in fact on semi-infinite arcs). In our examples, the descendant set of a vertex is a directed binary tree. Primitive highly arc transitive digraphs with finite out-valency are analysed in detail in [1] and [2]. It is shown that the descendant set of a vertex is quite constrained in such a digraph, in particular, there are only countably many possibilities for the descendant set. Thus whilst results in [1] suggest that there may be a possibility of classifying descendant sets of vertices in highly arc transitive primitive digraphs, our results indicate that there is no possibility

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of classifying the ones with a given isomorphism type of descendant set.

The methods we use follow closely those in [4]. However, we will keep this paper reasonably self-contained, and at times amplify on some of the arguments in [4].

We will use the folowing notation and terminology throughout the rest of the paper. We regard a digraph as a relational structure $\langle A; R^A\rangle$ where $R^A \subseteq A^2$ is a binary relation. So A (the domain of the structure) is the set of vertices and $R^A(a,b)$ means that $(a,b) \in R^A$ and there is a directed edge from a to b. Digraphs will be loopless and without multiple edges.

Now suppose $I \subseteq \mathbb{N}\backslash\{0,1,2\}$. An *$L_I$-structure* $\langle A; R^A, \{R_n^A : n \in I\}\rangle$with *domain* A is a relational structure where R^A is as above and for $n \in I$, $R_n^A \subseteq A^n$ is an n-ary relation (so if $I = \emptyset$, then the structure is just a digraph). An L_I-structure $\langle B; R^B, \{R_n^B : n \in I\}\rangle$ is a *substructure* of $\langle A; R^A, \{R_n^A : n \in I\}\rangle$ if $B \subseteq A$, $R^B = R^A \cap B^2$ and $R_n^B = R^A \cap B^n$ for all $n \in I$. An *isomorphism* between L_I-structures is a bijection which preserves all the relations (both ways); an *embedding* between L_I-structures is an injective map $B \to A$ which gives an isomorphism between B and the substructure on its image. Henceforth we freely confuse a structure with its domain (so refer to 'the L_I-structure A' rather than '$\langle A; R^A, \{R_n^A : n \in I\}\rangle$') and drop the superscript of R^A if it is clear from the context. We also write R_n instead of R_n^A if the context is clear. Given an L_I-structure A we refer to $\langle A; R^A\rangle$ as the *underlying digraph* of the structure and denote it by $A|_R$.

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2. The construction

For $I \subseteq \mathbb{N}\backslash\{0,1,2\}$ we will construct a countable L_I-structure N^I whose automorphism group is primitive, and whose underlying digraph $N^I|_R$ is unbalanced. We do this in such a way that if $I \neq J \subseteq \mathbb{N}\backslash\{0,1,2\}$ then the underlying digraphs $N^I|_R$ and $N^J|_R$ are non-isomorphic. This will establish the theorem (the automorphism group of the digraph contains the automorphism group of the L_I-structure, so is primitive). The construction follows [4]: we build N^I as the Fraïssé limit of a suitable amalgamation class $(\mathcal{C}_I, \leq^+)$.

2.1. **The amalgamation classes.** Denote the rooted (outward) directed binary tree by T. If A is an L_I-structure and $a \in A$ then the set of *descendants* of a in A, $desc_A(a)$ is the set of vertices (including a) in A that can be reached from a by an outward directed path, that is, $\{b : \exists a_1, \ldots, a_n \in A, (a, a_1), (a_1, a_2), \ldots, (a_n, b) \in R\}$. If X is a set of vertices in A, then write $desc_A(X) = \bigcup\{desc_A(x) : x \in X\}$. If $X = \{x_1, \ldots, x_n\}$ write $desc_A(x_1, \ldots, x_n)$ for $desc_A(X)$. Say that a set V of vertices of A is *finitely generated* with *generators* $a_1, \ldots, a_n$ if $V = desc_A(a_1, \ldots, a_n)$. The set of *ancestors* of a vertex $a \in A$ is the set of vertices $\{b \in A : a \in desc_A(b)\}$. If it is clear which structure we are working in then write $desc(a)$ for $desc_A(a)$ (in [4] the notation $a^{\Rightarrow}$ is used for $desc(a)$). We say that $d_1, \ldots, d_n \in A$ are *independent* (in A) if $desc(d_i) \cap desc(d_j) = \emptyset$ for $i \neq j$.

Definition 2.1. Let $A \subseteq B$ be L_I-structures. Say that A is *descendant closed* in B, written $A \leq B$ if for all $a \in A$, $desc_B(a) \subseteq A$. For $A \leq B$ and A finitely generated, define $A \leq^+ B$ to mean that for all $b \in B$, $desc(b) \cap A$ is finitely generated and if $desc(b) \backslash A$ is finite then $b \in A$.

Definition 2.2. Let $(\mathcal{C}_I, \leq^+)$ consist of countable L_I-structures A such that R gives a digraph on A and the following conditions hold:

(1) for all $a \in A$ we have that $desc(a)$ is isomorphic to T;
(2) for all $a \in A$, $desc(a) \leq^+ A$;
(3) A is finitely generated;
(4) if $a_1, \ldots, a_n \in A$ and $R_n^A(a_1, \ldots, a_n)$, then $a_1, \ldots a_n$ are independent in A, $desc(a_1, \ldots, a_n) \leq^+ A$, and $a_1, \ldots, a_n$ have no common ancestor in A;
(5) the number of instances of the relations R_n on A is finite (i.e. $\bigcup_{n \in I} R_n^A$ is a finite set).

Proposition 2.3 (Hereditary Property)**.** *If $A \in \mathcal{C}_I$ and $B \leq A$ is a finitely generated substructure of A, then $B \in \mathcal{C}_I$.*

Proof. This is straightforward to check. □

Lemma 2.4. *If $C \in \mathcal{C}_I$ and $A, B \leq C$ are finitely generated, then $A \cap B$ is finitely generated.*

Proof. Since A, B are finitely generated and $A, B \leq C$ we can write $A = \bigcup_{1 \leq i \leq n} desc_C(a_i)$ and $B = \bigcup_{1 \leq j \leq m} desc_C(b_j)$ for some $n, m \in \mathbb{N}$. Then we have

$$A \cap B = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \left(desc_C(a_i) \cap desc_C(b_j)\right).$$

By condition 2 of Definition 2.2 we have $desc_C(a_i) \cap desc_C(b_j)$ is finitely generated and so $A \cap B$ is finitely generated. □

The following lemma is also found in [4] (as Lemma 2.2 (iv)) so the proof is omitted. However, note that whilst the definition of $A \leq^+ B$ is the same, the definition of $A \leq B$ is different in this case.

Lemma 2.5 ([4], Lemma 2.2 (iv))**.** *Let $X \subseteq Y \subseteq Z \in \mathcal{C}_I$. If $X \leq^+ Y$ and $Y \leq^+ Z$ and X, Y are finitely generated, then $X \leq^+ Z$.* $\square$

We say that an embedding $\alpha : A \to B$ between structures in $\mathcal{C}_I$ is a $\leq^+$*-embedding* if $\alpha(A) \leq^+ B$.

Proposition 2.6 (Amalgamation Property)**.** *Suppose $A, B_1, B_2 \in \mathcal{C}_I$ and $\alpha_i : A \to B_i$ are $\leq^+$-embeddings. Then there is an L_I-structure $C \in \mathcal{C}_I$ and $\leq^+$-embeddings $\beta_i : B_i \to C$ such that $\beta_1\alpha_1 = \beta_2\alpha_2$.*

Proof. We may assume $B_1 \cap B_2 = A$ and α_i is the identity on B_i, so $A \leq^+ B_i$ for $i = 1, 2$. Let C be the *free amalgam* $B_1 \coprod_A B_2$. So C is the L_I-structure on the disjoint union $B_1 \cup B_2$ over A where the only relations are those induced from B_1 and B_2. We show that $C \in \mathcal{C}_I$ and $B_i \leq^+ C$, so we can let the β_i be the identity maps on the respective B_i. Note that by construction, $B_i \leq C$ and C is finitely generated.

To show that $C \in \mathcal{C}_I$ we need to check that the five conditions in Definition 2.2 hold in C. Conditions 1, 3 and 5 are satisfied directly from the construction of C. We next show that condition 4 holds. Since C is the free amalgam of B_1 and B_2 over A, if $R_n^C(c_1, \ldots, c_n)$ holds then without loss of generality $c_1, \ldots, c_n \in B_1$. Hence, $desc(c_i) \cap desc(c_j) = \emptyset$ for $i \neq j$ and $desc(c_1, \ldots, c_n) \leq^+ B_1$. Using Lemma 2.5 we find $desc(c_1, \ldots, c_n) \leq^+ C$. Finally, $c_1, \ldots, c_n$ have no common ancestor in B_1, so have no common ancestor in C (otherwise they have a common ancestor in $B_2 \setminus A$, and as C is a free amalgam this implies $c_1, \ldots, c_n \in A$, which then contradicts $R_n^{B_2}(c_1, \ldots, c_n)$). This gives 4.

For condition 2, suppose $b_1, b_2 \in C$. We need to show that $desc(b_1) \cap desc(b_2)$ is finitely generated and that if $desc(b_2) \setminus desc(b_1)$ is finite then $b_2 \in desc(b_1)$. If $b_1, b_2 \in B_i$ for some i then this is immediate as $B_i \leq C$ and $B_i \in \mathcal{C}_I$. So assume that $b_i \in B_i$. Then $desc(b_1) \cap desc(b_2) = (desc(b_1) \cap A) \cap (desc(b_2) \cap A)$. Each of $desc(b_i) \cap A$ is finitely generated as $A \leq^+ B_i$. Thus their intersection is finitely generated, by Lemma 2.4 applied in A. As $desc(b_2) \setminus desc(b_1) \supseteq desc(b_2) \setminus A$, if this is finite then $A \leq^+ B_2$ implies $b_2 \in A$, and so $b_2 \in desc(b_1)$ because $B_1 \in \mathcal{C}_I$. This gives condition 2, and so $C \in \mathcal{C}_I$.

Finally we show (without loss of generality) that $B_1 \leq^+ C$. As B_1 is finitely generated and $C \in \mathcal{C}_I$, Lemma 2.4 shows that $B_1 \cap desc(c)$ is finitely generated for all $c \in C$. Suppose that $c \in B_2$ and $desc(c) \setminus B_1$ is finite. Then as above $desc(c) \setminus A$ is finite and it follows that $c \in B_1$. $\square$

Let $A, A', B, B' \in \mathcal{C}_I$ and let $f : A \to B$, $f' : A' \to B'$ be $\leq^+$-embeddings. We say that f is *isomorphic* to f' if there exist isomorphisms $g : A \to A'$, $h : B \to B'$ such that $f'g = hf$.

Proposition 2.7. *There are countably many isomorphism types of $\leq^+$-embeddings in $(\mathcal{C}_I, \leq^+)$.*

Proof. Because each structure in $\mathcal{C}_I$ has only finitely many instances of the relations R_n, it is sufficient to show that there are countably many isomorphism types of $\leq^+$-embeddings between the underlying digraphs of elements of $\mathcal{C}_I$. In other words, it is sufficient to prove the Proposition in the case where $I = \emptyset$. This is done in Lemma 2.14 of [4], but we give a sketch of the argument here.

First we show by induction on $n \in \mathbb{N}$ that there are countably many isomorphism types of n-generator structures in $\mathcal{C}_\emptyset$. This is clear for $n = 1$. For the inductive step, note that if $C = desc(c_1, \ldots, c_n)$, then C is the free amalgam of $desc(c_1, \ldots, c_{n-1})$ and $desc(c_n)$ over their intersection A, and there are only countably many possibilities for the isomorphism type of the former. Moreover, A is finitely generated and $A = desc(a_1, \ldots, a_m)$ for some independent set $a_1, \ldots, a_m$, by definition of $\mathcal{C}_\emptyset$. There are only countably many possibilities for this independent set within $desc(c_1, \ldots, c_{n-1})$ and within $desc(c_n)$, and any automorphism of it extends to an automorphism of the latter. Thus there are only countably many possibilities for the isomorphism type of C.

We now prove the stronger statement that if $C, D \in \mathcal{C}_\emptyset$ then there are countably many isomorphism types of $\leq$-embeddings $\alpha : C \to D$. Again this is by induction on the number n of generators of C. Suppose $C = desc(c_1, \ldots, c_n)$ and by induction we may assume that $\alpha|desc(c_1, \ldots, c_{n-1})$ and $\alpha(c_n)$ are fixed. As in [4], it then suffices to observe that as $B \in \mathcal{C}_\emptyset$, there is some natural number k such that any automorphism of $desc(c_n)$ which fixes all vertices at out-distance at most k from c_n can be extended to an automorphism of B. For then there are only finitely many possibilities for the isomorphism type of α with the given $\alpha|desc(c_1, \ldots, c_{n-1})$ and $\alpha(c_n)$. □

The above propositions give us that the classes $(\mathcal{C}_I, \leq^+)$ defined in Definition 2.2 are *amalgamation classes.*

2.2. **The Fraïssé limits.**

Theorem 2.8. *There is a countable L_I-structure N^I such that*

(1) *N^I is the union of substructures $N_1 \subseteq N_2 \subseteq \ldots$ such that each $N_i \in \mathcal{C}_I$ and $N_i \leq^+ N_{i+1}$ for $i \in \mathbb{N}$;*
(2) *(Extension Property) whenever $A \leq^+ N_i$ is finitely generated and $\theta : A \to B \in \mathcal{C}_I$ is a $\leq^+$-embedding, there is $s \geq i$ and a $\leq^+$-embedding $f : B \to N_s$ with $f(\theta(a)) = a$ for all $a \in A$.*

Moreover, N^I is determined up to isomorphism by these two properties, and is $\leq^+$-homogeneous: if $A_1, A_2 \leq^+ N_i$ are finitely generated and $f : A_1 \to A_2$ is an isomorphism, then f can be extended to an automorphism of N^I.

We refer to N^I in the above as the *Fraïssé limit* of the amalgamation class $(\mathcal{C}_I, \leq^+)$. For finitely generated $A \leq N^I$ we write $A \leq^+ N^I$ to indicate that $A \leq^+ N_i$ for some i (this depends only on N^I).

Proof. The proof is standard, as in [3]. We construct the N_i inductively, taking $N_1 = \emptyset$, for example. For the purposes of the proof it will be useful to fix a bijection $\eta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the property that $\eta(a, b, c) \geq a, b, c$.

Suppose that we have constructed $N_1 \leq^+ \ldots \leq^+ N_i \in \mathcal{C}_I$. There are countably many finitely generated $\leq^+$-substructures of N_i, so we can list these as $(A^i_j : j \in \mathbb{N})$. For each A^i_j there are countably many isomorphism types of $\leq^+$-embeddings into elements of $\mathcal{C}_I$: list these as $\theta^i_{jk} : A^i_j \to B_k$. Note that at stage i we will have done this for each N_m with $m \leq i$. The point is that the extension problem (as in Property 2) corresponding to θ^i_{jk} will be solved at stage $s = \eta(i, j, k) + 1$. So let $(i', j', k') = \eta^{-1}(i)$. We have $\theta^{i'}_{j'k'} : A^{i'}_{j'} \to B_{k'}$ and $A^{i'}_{j'} \leq^+ N_{i'} \leq^+ N_i$. Then use the amalgamation property of $\mathcal{C}_I$ on $A^{i'}_{j'}$, $B_{k'}$ and N_i to get $N_{i+1} \in \mathcal{C}_I$ such that $N_i \leq^+ N_{i+1}$ and $B_{k'} \leq^+ N_{i+1}$. This completes the inductive construction of the N_j with properties 1 and 2.

The proof of the 'Moreover' part is a standard back-and-forth argument. Suppose $(\tilde{N}^I; \tilde{N}_i)$ also satisfy properties 1 and 2. Suppose $A \leq N^I$ and $\tilde{A} \leq^+ \tilde{N}^I$ are finitely generated and $f : A \to \tilde{A}$ is an isomorphism. By countability, and by symmetry, it is enough to show that if $b \in N^I$ then there exist finitely generated $B \leq N^I$ and $\tilde{B} \leq^+ \tilde{N}^I$, containing A and $\tilde{A}$ respectively, with $b \in B$, and an isomorphism $g : B \to \tilde{B}$ extending f. Using property 1 in N^I we can find a finitely generated $B \leq^+ N^I$ containing A and b. Using property 2 in $\tilde{N}^I$, we can extend f to a $\leq^+$-embedding of B into N^I. $\square$

2.3. **Primitivity of** $Aut(N^I)$**.** We now prove some properties of the Fraïssé limit of $(\mathcal{C}_I, \leq^+)$ and show that $Aut(N^I)$ is primitive.

If $A \in \mathcal{C}_I$ and $X \subseteq A$, define the *closure* of X in A to be $cl^A(X) = \{y \in A : desc_A(y) \backslash desc_A(X) \text{ is finite}\}$. It is clear that if $X \subseteq Y \leq^+ A$ then $cl^A(X) \subseteq Y$ and $desc(X) \leq cl^A(X) \leq A$. The following shows that if X is finite then $cl^A(X)$ is the smallest $\leq^+$-subset of A which contains X.

Lemma 2.9. *Suppose $A \in \mathcal{C}_I$ and $X \subseteq A$ is finite. Then $cl^A(X)$ is finitely generated and $cl^A(X) \leq^+ A$.*

Proof. By Lemma 2.4 we only need to prove that $cl^A(X)$ is finitely generated. Let $Y = desc_A(X)$. We claim that $cl^A(X) \setminus Y$ is finite. If $b \in A$ then $desc(b) \cap Y$ is finitely generated, equal to $desc_A(c_1, \ldots, c_k)$, say. If $c \in desc_A(b) \cap (cl^A(X) \setminus Y))$ then $c_i \in desc(c)$, for some i. But there are only finitely many possibilities for such a c in $desc(b)$, as the latter is a directed, rooted tree. Thus $desc(b) \cap (cl^A(X) \setminus Y))$ is finite. As A is finitely generated, we obtain the claim, and hence the Lemma. □

Note that we can make the same definition for $cl_{N^I}(X)$ when X is a finite subset of N^I, and the above lemma also holds (by property 1 of N^I).

Lemma 2.10. *The digraph $N^I|_R$ is connected.*

Proof. Let $a_1, a_2 \in N^I$. If $desc(a_1) \cap desc(a_2) \neq \emptyset$ then there is an undirected path from a_1 to a_2 going via this intersection. So suppose a_1, a_2 are independent and let $A = cl(a_1, a_2)$. The structure consisting of the free amalgam of A and a binary tree T over $\emptyset$ is in $\mathcal{C}_I$ so using the extension property over A there exists $b \in N^I$ such that $desc(b) \cup A \leq^+ N^I$ and $desc(b) \cap A = \emptyset$. Then $desc(a_i) \cap desc(b) = \emptyset$ and $desc(a_i) \cup desc(b) \leq^+ N^I$. Therefore by the extension property there exist $c_i \in N^I$ such that $a_i, b \in desc(c_i)$. In particular, a_1 and a_2 are joined by an undirected path in $N^I|_R$. □

Proposition 2.11. *The automorphism group $Aut(N^I)$ is primitive on N^I.*

Proof. This is similar to the proof of ([4], Theorem 2.9), though we offer a different argument in Case 3 below as the original argument appears to be somewhat inaccurate. We see that $Aut(N^I)$ is transitive on N^I due to its $\leq^+$-homogeneity and by conditions 1 and 2 in Definition 2.2. Suppose $a \neq b \in N^I$ and consider the *orbital graph* G with vertex set the elements of N^I and edge set $E = \{\{fa, fb\} : f \in Aut(N^I)\}$. By the criterion of D. G. Higman (from [5]) it will suffice to show that all such G are connected. As N^I is connected via R-edges by Lemma 2.10, it is enough to show that if $x, y \in N^I$ are such that (x, y) is an R-edge of N^I then x and y lie in the same connected component of G. Without loss, we can assume $x = a$. Let $H_1 = cl_{N^I}(a, b)$.

Case 1: Suppose $desc(a) \cap desc(b) = \emptyset$. Let H_2 be a copy of H_1 with $a', b' \in H_2$ corresponding to $a, b \in H_1$. Recalling that y is an out-vertex of a, identify $desc_{H_1}(y)$ with $desc_{H_2}(b)$, and take the free amalgam $H_{1,2}$ over $desc_{H_1}(y)$ of H_1 and H_2. It is easy to see that $desc(a') \cup desc(b) \leq^+ H_{1,2}$, so we can adjoin a finite set X of new vertices to $H_{1,2}$ to obtain a structure $P \supseteq H_{1,2}$ in which $H_3 = cl^P(a', b) = desc(a', b) \cup X$ is isomorphic to H_1 (via an isomorphism taking a' to a and b to b). So P is the union of H_1, H_2 and H_3, and $H_1 \cap H_3 = desc(b)$, $H_3 \cap H_2 = desc(a')$ and $H_1 \cap H_2 = desc(y)$. Moreover, any edge (and any R_n relation) is contained entirely within some H_i.

Claim. $P \in \mathcal{C}_I$.

Proof of Claim. It is clear by the construction of P that conditions 1, 3 and 5 in Definition 2.2 hold. For condition 2 note that each

H_i is descendant closed in P. We see that $desc(y) \leq^+ H_1$ and also $desc(y) \leq^+ H_2$, so by the amalgamation lemma $H_1 \leq^+ H_{1,2}$. As $desc(b) \cup desc(a') \leq^+ H_{1,2}$ and P is the free amalgam of $H_{1,2}$ and H_3 over this, we get that $H_3 \leq^+ P$.

This argument can be seen to be symmetrical in $1, 2, 3$ (where $H_{i,j}$ is the union of H_i and H_j: note that these are freely amalgamated over their intersection in P). So we have $H_i \leq^+ P$ for $i = 1, 2, 3$. Then this gives us $desc(p) \leq^+ P$ for every $p \in P$, that is, condition 2 holds.

Finally, suppose $R_n^P(p_1, \ldots, p_n)$ for some $p_1, \ldots, p_n \in P$. By the construction of P this implies that $p_1, \ldots, p_n \in H_i$ for some i. Therefore $R_n^{H_i}(p_1, \ldots, p_n)$ and so the p_j are independent, and $desc(p_1, \ldots, p_n) \leq^+ H_i$ and hence by Lemma 2.5, $desc(p_1, \ldots, p_n) \leq^+ P$. Suppose for a contradiction that $p_1, \ldots, p_n$ have a common ancestor, say $q \in P$. So $q \in H_j$ for some $j \neq i$. But H_i and H_j are freely amalgamated over their intersection and as this is the descendent set of a single point, not all of $p_1, \ldots, p_n$ are in the intersection. As $H_i \cap H_j \leq P$, we have $q \notin H_i \cap H_j$. But this contradicts freeness of the amalgamation. Therefore condition 4 holds and so we have $P \in \mathcal{C}_I$.

$\square$ Claim.

Now we use the extension property to obtain a $\leq^+$-embedding $\phi : P \to N^I$ which is the identity on H_1. By $\leq^+$-homogeneity and the construction of P, we have that $a, \phi(b), \phi(a'), y$ is a path in the orbital graph G. In particular, $x = a$ and y are in the same connected component of G.

Case 2: Suppose that $b \in desc(a)$. In this case let b_0 denote the predecessor of b in $desc(a)$, so (b_0, b) is an R-edge in N^I. Let $b_1 \in desc(a)$ be the other successor of b_0. Then there is an automorphism of N^I fixing a and interchanging b and b_1. So b and b_1 are connected in the orbital graph G. We have $desc(b) \cap desc(b_1) = \emptyset$ and hence case 1 gives that the orbital graph with $\{b, b_1\}$ as an edge is connected. Therefore the orbital graph G is also connected.

By condition 2 in Definition 2.2, the only remaining case is:
Case 3: Suppose $desc(b) \backslash desc(a)$ and $desc(a) \backslash desc(b)$ are infinite. In this case let $x_1, \ldots, x_r$ be a minimal generating set for $desc(a) \cap desc(b)$. Thus $x_1, \ldots, x_r$ are independent, and we prove that the orbital graph G is connected in this case by induction on r, taking $r = 0$ as the base case (given by case 1 above). We can assume that x_r is at maximal distance from a, amongst the x_i. Let z be the immediate predecessor of x_r in $desc(a)$. Note that $z \notin desc(a) \cap desc(b)$ by minimality of the generating set. As $desc(a) \cap desc(b) \leq^+ desc(a)$, not all of the successors of z lie in $desc(a) \cap desc(b)$. So we can choose x'_r to be one of its successors which is not amongst $x_1, \ldots, x_r$. The distance of x'_r from a in $desc(a)$ is no

smaller than the distance of that of any of the x_i. Thus $x_1, \dots, x_{r-1}, x'_r$ is independent and $desc(x_1, \dots, x_{r-1}, x'_r) \leq^+ desc(a)$.

By a free amalgamation and the extension property there is $b_1 \in N^I$ such that $desc(b_1) \cap cl(a, b) = desc(x_1, \dots, x_{r-1}, x'_r)$ and there exists an isomorphism $f : cl(a, b) \to cl(a, b_1)$ with $f(a, b, x_1, \dots, x_{r-1}, x_r) = (a, b_1, x_1, \dots, x_{r-1}, x'_r)$. By $\leq^+$-homogeneity, this extends to an automorphism of N^I. Thus b and b_1 are in the same connected component of the orbital graph G. But $desc(b) \cap desc(b_1) = desc(x_1, \dots, x_{r-1})$, and by the induction hypothesis the orbital graph with $\{b, b_1\}$ as an edge is connected. Thus G is connected. □

2.4. **Non-isomorphism of the underlying digraphs.** Recall that if $I \subseteq \mathbb{N} \setminus \{0, 1, 2\}$ then $N^I|_R$ denotes the underlying digraph of the L_I-structure N^I: thus we are forgetting about the relations R_n. We show that different choices of I give non-isomorphic digraphs.

Proposition 2.12. *Let $n \neq 0, 1, 2$ be a natural number. Then $n \in I$ if and only if there exist $a_1, \dots, a_n \in N^I|_R$ with the following properties:*

(1) *$a_1, \dots, a_n$ are independent and $A = desc(a_1, \dots, a_n) \leq^+ N^I|_R$;*
(2) *$a_1, \dots, a_n$ have no common ancestor in $N^I|_R$;*
(3) *every finite subset X of A with $cl^A(X) \neq A$ has a common ancestor in $N^I|_R$.*

Proof. First suppose that $n \in I$. Then there exist $a_1, \dots, a_n \in N^I$ with $R_n(a_1, \dots, a_n)$ holding, and such that $R_n(a_1, \dots, a_n)$ is the only instance of a relation R_m which holds on $A = desc(a_1, \dots, a_n)$ (simply because this structure is in $\mathcal{C}_I$). So conditions 1 and 2 above hold. Now let X be a finite subset of A with $Y = cl^A(X) \neq A$. We can assume that $X = \{x_1, \dots, x_r\}$ is a minimal generating set for Y. As A is just n disjoint copies of T, the set $\{x_1, \dots, x_r\}$ is independent. Note that there are no instances of relations R_m on Y. We can therefore find a copy of the structure Y as a $\leq^+$-substructure of the tree T (it is simply r disjoint copies of T). By the extension property it follows that there exists $c \in N^I$ with $desc(c) \supseteq Y$. In particular, c is a common ancestor of the elements of X, so condition 3 also holds.

Now suppose that $a_1, \dots, a_n$ have the given properties $1, 2, 3$ and, for a contradiction, $n \notin I$. Then there is no relationship between the points of A except digraph relations. To see this let $a'_1, \dots, a'_k \in A$ and suppose $R^A_k(a'_1, \dots, a'_k)$. So of course $k \neq n$. Then by condition 4 of Definition 2.2 we must have $a'_1, \dots, a'_k$ independent and $desc(a'_1, \dots, a'_k) \leq^+ A$. As $k \neq n$ we have $cl^A(a'_1, \dots, a'_k) \neq A$. Hence, by condition 3 $a'_1, \dots, a'_k$ have a common ancestor in $N^I|_R$, which contradicts $R^A_k(a'_1, \dots, a'_k)$. But we can now use the same argument as in the previous paragraph to show that there is some $c \in N^I$ with $A \subseteq desc(c)$. This gives a common ancestor of $a_1, \dots, a_n$ in N^I, which contradicts property 2 of $a_1, \dots, a_n$. □

Proposition 2.13. *If $I \neq J$ are subsets of $\mathbb{N}\backslash\{0,1,2\}$ then the digraphs $N^I|_R$ and $N^J|_R$ are not isomorphic.*

Proof. This follows immediately from Proposition 2.12: properties 1, 2, 3 there all relate to the digraph $N^I|_R$ and allow us to recover I from its structure. □

We therefore have our main result:

Theorem 2.14. *There are continuum many pairwise non-isomorphic countable highly arc transitive directed graphs in which each vertex has finite out-valency and infinite in-valency, and whose automorphism group is primitive.*

Proof. As any automorphism of N^I is a digraph automorphism, the digraphs $N^I|_R$ are certainly primitive and highly arc transitive. The previous proposition shows that the continuum-many possible choices for I result in non-isomorphic digraphs. □

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