An introduction to ampleness

David M. Evans

0. Introduction

A conjecture of Zilber from around 1980 asserted that the 'classical' examples of strongly minimal structures – pure set; vector spaces; algebraically closed fields – are 'essentially' the only examples of strongly minimal structures. (Note that by the Cherlin - Zilber Theorem, this is true for ω-categorical structures.) More precisely, the conjecture could be stated as saying that a strongly minimal structure which is not one-based interprets an infinite field.

This conjecture was refuted by Ehud Hrushovski in 1988 (in an unpublished manuscript which was incorporated into [**9**]). Using a method now described as 'the Hrushovski construction' or 'Hrushovski's predimension construction', he produced a strongly minimal structure which is not one-based, but which does not interpret an infinite group. He also observed that these structures have some properties - flatness, CM-triviality - which restrict the complexity of forking.

In [**14**], Pillay proposed a hierarchy of properties of forking in stable theories:

$$1\text{-ample} \Leftarrow 2\text{-ample} \Leftarrow \ldots \Leftarrow n\text{-ample} \Leftarrow \ldots$$

where

$$1\text{-ample} = \text{not one-based}$$

$$2\text{-ample} = \text{not CM-trivial.}$$

So Hrushovski's constructions are 1-ample but not 2-ample and Pillay showed that a stable field is n-ample for all $n \in \mathbb{N}$.

It is an open problem whether there is a strongly minimal structure which is 2-ample and does not interpret an infinite field.

Outside the finite rank context, another restriction on the complexity of forking becomes relevant: the notion of *triviality*, discussed in [**7**]. A structure of finite Morley rank where forking is trivial is necessarily one-based and it is easy to see that a trivial stable structure cannot interpret an infinite group. The *free pseudoplane* is an example of an ω-stable structure (of infinite rank) which is trivial and not one-based - in fact it is 1-ample and not 2-ample. Baudisch and Pillay [**2**] constructed an ω-stable structure - a 'free pseudospace' - which is 2-ample and trivial. This has recently been extended by Tent [**17**] and Baudisch, Martin-Pizarro and Ziegler [**3**] to obtain 'free n-spaces' which are ω-stable, trivial and n-ample, but not $n+1$-ample.

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A non-trivial, 2-ample, ω-stable structure can easily be obtained by taking the disjoint union of the free pseudospace with a Hrushovski construction, so care has to be taken in asking about how to remove the triviality in these constructions.

The plan of the talks is:

(1) The basic Hrushovski construction (the infinite rank version): description of types and forking; weak elimination of imaginaries.
(2) Complexity of forking in stable theories: one-basedness, CM-triviality and ampleness. Triviality. Examples and behaviour under reducts.
(3) Obtaining ampleness: the free pseudospace and other constructions.

1. Hrushovski constructions

The ultimate aim of the construction in [**9**] is to build a strongly minimal set: a structure with a dimension on it. We build it from finite structures each of which carries in a natural way a dimension. We will not describe the whole construction here: what we do will produce a structure of infinite Morley rank. To obtain a structure of finite Morley rank, the method needs an extra ingredient, usually referred to as 'collapse.' More thorough references include [**20**] and [**1**]. The construction is also discussed in [**18**].

1.1. Predimension and dimension. Suppose k, ℓ, r are given natural numbers. We consider a language L having just a r-ary relation symbol R. We work with L-structures which are models of some $\forall$-theory T' (which will be specified further as we go along).

If $B \models T'$ is finite define the *predimension* of B to be

$$\delta(B) = \ell|B| - k|R^B|.$$

If $B \subseteq A \models T'$ we can regard B as a substructure of A and if B is finite we can consider $\delta(B)$. We let $\bar{\mathcal{C}}_0$ be the class of structures $A \models T'$ with the property that

$$\delta(B) \geq 0 \text{ for all finite } B \subseteq A.$$

Let $\mathcal{C}_0$ be the finite structures in $\bar{\mathcal{C}}_0$.

It is easy to see that there is a universal L-theory T_0 whose models are precisely the structures in $\bar{\mathcal{C}}_0$.

Examples 1.1. Good examples to have in mind are:

(i) (from [**9**]) $r = 3$, $k = \ell = 1$ and T' says:

$$(\forall x_1 x_2 x_3)(R(x_1, x_2, x_3) \to \bigwedge_{i \neq j}(x_i \neq x_j))$$

and for every permutation π of $1, 2, 3$:

$$(\forall x_1 x_2 x_3)(R(x_1, x_2, x_3) \leftrightarrow R(x_{\pi 1}, x_{\pi 2}, x_{\pi 3})).$$

Thus in a model A of these, we can regard the interpretation R^A of R as a set of 3-subsets of A (rather than a set of triples).

(ii) $r = 2$, $\ell = 2$, $k = 1$ and T' is the theory of graphs.

Lemma 1.2 (Submodularity). *If $A \in \bar{\mathcal{C}}_0$ and B, C are finite subsets of A, then*

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

There is equality iff $R^{B\cup C} = R^B \cup R^C$ (in which case we say that B, C are freely amalgamated over their intersection).

Proof. Note that the left-hand side minus the right-hand side of the inequality is:

$$\ell(|B \cup C| - (|B| + |C| - |B \cap C|)) - k(|R^{B\cup C}| - |R^B| - |R^C| + |R^{B\cap C}|).$$

As $R^{B\cap C} = R^B \cap R^C$, this is equal to

$$-k(|R^{B\cup C}| - |R^B \cup R^C|).$$

Hence the result. □

If $A \subseteq B \in \bar{\mathcal{C}}_0$ is finite and for all finite B' with $A \subseteq B' \subseteq B$ we have $\delta(A) \leq \delta(B')$, then we say that A is *self-sufficient* in B and write $A \leq B$.

Lemma 1.3. *Suppose $B \in \mathcal{C}_0$.*

1. *If $A \leq B$ and $X \subseteq B$, then $A \cap X \leq X$.*
2. *If $A \leq B$ and $B \leq C \in \mathcal{C}_0$ then $A \leq C$.*
3. *If $A_1, A_2 \leq B$ then $A_1 \cap A_2 \leq B$.*

Proof. (1) Let $A \cap X \subseteq Y \subseteq X$. Then

$$\delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(Y \cap A).$$

So as $A \cap X = A \cap Y$ we have:

$$\delta(Y) - \delta(X \cap A) \geq \delta(A \cup Y) - \delta(A) \geq 0.$$

(2) Let $A \subseteq X \subseteq C$. As $B \leq C$ we have $X \cap B \leq X$ (by (1)) so $\delta(X \cap B) \leq \delta(X)$. Also, $A \subseteq X \cap B \subseteq B$ so $\delta(A) \leq \delta(X \cap B)$, by $A \leq B$. So $\delta(A) \leq \delta(X)$.

(3) By (1) we have $A_1 \cap A_2 \leq A_2$. So $A_1 \cap A_2 \leq B$, using (2). □

If $B \in \bar{\mathcal{C}}_0$ and $A \subseteq B$ then we write $A \leq B$ when $A \cap X \leq X$ for all finite $X \subseteq B$. It can be checked that the above lemma still holds.

If X is a finite subset of $B \in \bar{\mathcal{C}}_0$ then there is a finite set C with $X \subseteq C \subseteq B$ and $\delta(C)$ as small as possible. Then by definition, $C \leq B$. Now take C as small as possible. By (3) of the above lemma, C is uniquely determined by X: it is the intersection of all self-sufficient subsets of B which contain X. We refer to this as the *self-sufficient closure* of X in B and denote it by $\mathrm{cl}^{\leq}_B(X)$. Write $d_B(X) = \delta(\mathrm{cl}^{\leq}_B(X))$. By the above discussion:

$$d_B(X) = \min\{\delta(C) : X \subseteq C \subseteq_f B\}.$$

(Where $C \subseteq_f B$ means C is a finite subset of B.)

This is the *dimension* of X in B. It is clear that if $X \subseteq Y \subseteq_f B$ then $d_B(X) \leq d_B(Y)$.

Exercise: Show that self-sufficient closure is a closure operation, but that it does not necessarily satisfy the exchange property.

Lemma 1.4. *If X, Y are finite subsets of $B \in \bar{\mathcal{C}}_0$ then*

$$d_B(X \cup Y) \leq d_B(X) + d_B(Y) - d_B(X \cap Y).$$

Proof. Let X', Y' be the self-sufficient closures of X and Y in B. Then

$$d_B(X \cup Y) = d_B(X' \cup Y') \leq \delta(X' \cup Y') \leq \delta(X') + \delta(Y') - \delta(X' \cap Y').$$

Now, $X \cap Y \subseteq X' \cap Y'$ and the latter is self-sufficient in B. So $d_B(X \cap Y) \leq d_B(X' \cap Y') = \delta(X' \cap Y')$. The result follows. □

Remarks 1.5. From the proof we can read off when we have equality in the above. Suppose for simplicity that X, Y are self-sufficient. Then there is equality in the lemma iff $X \cup Y \leq B$ and X, Y are freely amalgamated over their intersection.

We now relativise the dimenson function. Suppose $B \in \bar{\mathcal{C}}_0$ and $\bar{a}$ is a tuple of elements in B and C a finite subset of B. Define the dimension of $\bar{a}$ over C to be:

$$d_B(\bar{a}/C) = d_B(\bar{a}C) - d_B(C).$$

(Where $\bar{a}C$ denotes the union of C and the elements in $\bar{a}$.)

Lemma 1.6. *If $\bar{a}$, $\bar{b}$ are tuples in $B \in \bar{\mathcal{C}}_0$ and C is a finite subset of B then:*

1. $d_B(\bar{a}\bar{b}/C) = d_B(\bar{a}/\bar{b}C) + d_B(\bar{b}/C)$.
2. $d_B(\bar{a}\bar{b}/C) \leq d_B(\bar{a}/C) + d_B(\bar{b}/C)$.
3. *If $C' \subseteq C$ then* $d_B(\bar{a}/C') \geq d_B(\bar{a}/C)$.

Proof. Drop the subscript B here. (1) is by definition and (2) follows from (1) and (3).
To prove (3), let $A' = \mathrm{cl}^{\leq}(\bar{a}C')$. Then

$$d(\bar{a}C) = d(A' \cup C) \leq d(A') + d(C) - d(A' \cap C) \leq d(A') + d(C) - d(C').$$

Rearranging gives what we want. □

We can extend this to arbitrary $C \subseteq B$. We define $d_B(\bar{a}/C)$ to be the minimum of $d(\bar{a}/C')$ for $C' \subseteq_f C$. By (3), this is harmless if C is actually finite. It can then be shown that the above lemma holds for arbitrary C.

1.2. Amalgamation. Suppose B_1, B_2 are structures in $\bar{\mathcal{C}}_0$ (or indeed, just models of T') with a common substructure A. We can assume without loss of generality that $A = B_1 \cap B_2$. We form another structure E with domain $E = B_1 \cup B_2$ and relations $R^E = R^{B_1} \cup R^{B_2}$. We refer to this as the *free amalgam* of B_1 and B_2 over A. Henceforth **assume that the class of models of T' is closed under free amalgamation**.

Lemma 1.7 (Free amalgamation lemma). *Suppose $B_1, B_2 \in \bar{\mathcal{C}}_0$ have a common substructure A. Suppose that $A \leq B_1$. Then the free amalgam E of B_1 and B_2 over A is in $\bar{\mathcal{C}}_0$ and $B_2 \leq E$.*

Proof. Note that the condition of being in $\bar{\mathcal{C}}_0$ is equivalent to the empty set being self-sufficient. So it suffices to prove $B_2 \leq E$, for then $\emptyset \leq B_2 \leq E$, and $\emptyset \leq E$ follows.

Let X be a finite subset of E. Write $X_i = X \cap B_i$ and $X_0 = X \cap A$. We want to show that $X_2 \leq X$, so let $X_2 \subseteq Y \subseteq X$. Now, X is the free amalgam of X_1 and X_2 over X_0 so Y is the free amalgam over X_0 of X_2 and $Y \cap X_1$, whence $\delta(Y) = \delta(Y \cap X_1) + \delta(X_2) - \delta(X_0)$. Thus

$$\delta(Y) - \delta(X_2) = \delta(Y \cap X_1) - \delta(X_0).$$

As $A \leq B_1$ we have $X_0 \leq X_1$. So as $X_0 \subseteq Y \cap X_1 \subseteq X_1$, the above is ≥ 0. Thus $\delta(Y) \geq \delta(X_2)$, as required. □

Of course, if $A \leq B_2$ here, then we also obtain $B_1 \leq E$ (by symmetry of the argument). However, we can usefully obtain something sightly stronger.

Suppose m is a natural number. Let $Y \subseteq Z \in \bar{\mathcal{C}}_0$. Write $Y \leq^m Z$ to mean that $\delta(Y) \leq \delta(Z')$ whenever $Y \subseteq Z' \subseteq Z$ and $|Z' \setminus Y| \leq m$. It is easy to check that Lemma 1.3 holds with $\leq$ replaced by $\leq^m$ throughout (the same proof works).

LEMMA 1.8 (Strong free amalgamation lemma). *Suppose $B_1, B_2 \in \bar{\mathcal{C}}_0$ have a common substructure A. Suppose that $A \leq^m B_1$ and $A \leq B_2$. Then the free amalgam E of B_1 and B_2 over A is in $\bar{\mathcal{C}}_0$ and $B_2 \leq^m E$ and $B_1 \leq E$.*

PROOF. By the previous lemma we have $\emptyset \leq B_1 \leq E$ and so $E \in \bar{\mathcal{C}}_0$. The proof that $B_2 \leq^m E$ just requires careful inspection of the above proof. □

1.3. The uncollapsed generic.

THEOREM 1.9 (The generic structure for $(\mathcal{C}_0; \leq)$). *There is a countable $\mathcal{M} \in \bar{\mathcal{C}}_0$ satisfying the following properties:*

(C1): *$\mathcal{M}$ is the union of a chain of finite substructures*
$B_1 \leq B_2 \leq B_3 \leq \ldots$ all of which are in $\mathcal{C}_0$.

(C2): *If $A \leq \mathcal{M}$ is finite and $A \leq B \in \mathcal{C}_0$, then there is an embedding $f : B \to \mathcal{M}$ with $f(B) \leq \mathcal{M}$ and which is the identity on A.*

Moreover $\mathcal{M}$ is uniquely determined up to isomorphism by these two properties and is $\leq$-homogeneous (meaning: any isomorphism between finite self-sufficient substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

PROOF. *The construction:* First, note that any countable structure in $\bar{\mathcal{C}}_0$ satisfies (C1). To achieve C2, we construct the B_i inductively so that the following (which is equivalent to C2) holds:

($C2'$) If $A \leq B_i$ and $A \leq B \in \mathcal{C}_0$ then there is $j \geq i$ and a $\leq$-embedding $f : B \to B_j$ which is the identity on A.

Note that there are countably many isomorphism types of $A \leq B$ in $\mathcal{C}_0$. A standard 'organisational' trick allows us to show that we can just do one instance of the problem in ($C2'$). But this is what amalgamation does for us: we have $A \leq B_i$ and $A \leq B$ so let B_{i+1} be the free amalgam of B_i and B over A. Then $B_i \leq B_{i+1}$ and $B \leq B_{i+1}$.

Uniqueness: Suppose $\mathcal{M}$ and $\mathcal{M}'$ satisfy these properties. One shows that the set of isomorphisms $A \to A'$ where $A \leq \mathcal{M}$ and $A' \leq \mathcal{M}'$ are finite is a back-and-forth system. The 'moreover' part follows. □

The structure $\mathcal{M}$ is referred to as the *generic structure* for the *amalgamation class* $(\mathcal{C}_0; \leq)$.

We want to understand $Th(\mathcal{M})$.

1.4. Model theory of $\mathcal{M}$. We want to axiomatize $Th(\mathcal{M})$ and understand types. Recall that T_0 is the set of axioms for the class $\bar{\mathcal{C}}_0$, and $(C1)$ holds in any countable model of these.

The condition in $(C2)$ is not (*a priori*) first-order: how can we express 'for all $A \leq \mathcal{M}$' and '$f(B) \leq \mathcal{M}$? The trick is to replace $\leq$ here by the approximations $\leq^m$.

Note that for each m, and each n-tuple of variables $\bar{x}$ there is a formula $\psi_{m,n}(\bar{x})$ with the property that for every $C \in \bar{\mathcal{C}}_0$ and n-tuple $\bar{a}$ in C we have:

$$\bar{a} \leq^m C \Leftrightarrow C \models \psi_{m,n}(\bar{a}).$$

Suppose $A \leq B \in \mathcal{C}_0$. Let $\bar{x}$, $\bar{y}$ be tuples of variables with $\bar{x}$ corresponding to the distinct elements of A and $\bar{y}$ corresponding to the distinct elements of $B \setminus A$. Let $D_A(\bar{x})$ and $D_{A,B}(\bar{x},\bar{y})$ denote the basic diagrams of A and B respectively. Suppose A, B are of size n, k respectively. For each m let $\sigma^m_{A,B}$ be the closed L-formula:

$$\forall \bar{x} \exists \bar{y} (D_A(\bar{x}) \wedge \psi_{m,n}(\bar{x}) \to D_{A,B}(\bar{x},\bar{y}) \wedge \psi_{m,k}(\bar{x},\bar{y})).$$

Let T consist of T_0 together with these $\sigma^m_{A,B}$.

Theorem 1.10. *We have that $\mathcal{M} \models T$ and T is complete. Moreover, n-tuples $\bar{c}_1$, $\bar{c}_2$ in models $\mathcal{M}_1, \mathcal{M}_2$ of T have the same type iff $\bar{c}_1 \mapsto \bar{c}_2$ extends to an isomorphism between $\mathrm{cl}^{\leq}_{\mathcal{M}_1}(\bar{c}_1)$ and $\mathrm{cl}^{\leq}_{\mathcal{M}_2}(\bar{c}_2)$.*

Proof. *Step 1: $\mathcal{M} \models T$.*
We show $\mathcal{M} \models \sigma^m_{A,B}$. So suppose $A' \leq^m \mathcal{M}$ is isomorphic to A. We have to find $B' \leq^m \mathcal{M}$ isomorphic to B (over A). Let $C = \mathrm{cl}^{\leq}_{\mathcal{M}}(A')$. Let E be the free amalgam of C and B over A (which we identify with A'), and use Lemma 1.8. Then $C \leq E$, so we can use $(C2)$ in $\mathcal{M}$ to get a $\leq$-embedding $f : E \to \mathcal{M}$ which is the identity on C. Then $A' \leq fB \leq^m fE \leq \mathcal{M}$: so $B' = fB$ is what we want.

Step 2: If $\mathcal{N} \models T$ is ω-saturated, then $\mathcal{N}$ satisfies $(C2)$.
Suppose $A \leq \mathcal{N}$ is finite and $A \leq B$. Let $\bar{a}$ enumerate A and $n = |B|$. By the $\sigma^m_{A,B}$, (and compactness) the collection of formulas $\{D_{A,B}(\bar{a},\bar{y}) \wedge \psi_{m,n}(\bar{a}\bar{y}) : m < \omega\}$ is consistent. So as $\mathcal{N}$ is ω-saturated we get $\bar{b}$ in $\mathcal{N}$ which satisfies all of them. Then $\bar{a}\bar{b} \leq \mathcal{N}$ and this gives what we want.

It then follows easily that if $\mathcal{N}_1$, $\mathcal{N}_2$ are ω-saturated models of T, then the set of isomorphisms between finite $\leq$-substructures of $\mathcal{N}_1$ and $\mathcal{N}_2$ is a back-and-forth system (– we also need to know that any finite subset is contained in a finite $\leq$-subset). This gives the 'if' direction in the statement. For the converse, note that if two tuples have the same type, then so do their self-sufficient closures (as these are part of the algebraic closure). $\square$

1.5. ω-stability of $\mathcal{M}$. $\mathcal{M}$ is the generic structure for $(\mathcal{C}_0; \leq)$ as in the previous section and we will let $T = Th(\mathcal{M})$ (this is a harmless change of notation).

Suppose $\mathcal{M}' \models T$ is ω-saturated $B \leq \mathcal{M}'$ and $\bar{a}$ is a tuple in $\mathcal{M}'$. There is a finite $C \leq B$ with $d(\bar{a}/B) = d(\bar{a}/C)$ and we can assume that $\mathrm{cl}(\bar{a}C) \cap B = C$ (– if not, replace C by this intersection).

Claim: $\mathrm{cl}^{\leq}(\bar{a}C) \cup B \leq \mathcal{M}'$ and is the free amalgam of $\mathrm{cl}^{\leq}(\bar{a}C)$ and B over C.

Proof of Claim: Let $A = \mathrm{cl}(\bar{a}C)$. It suffices to prove the claim when B is finite (– by considering finite closed subsets of the original B). By definition of δ if A, B are not freely amalgamated over C then $\delta(\mathrm{cl}^{\leq}(\bar{a}B)) \leq \delta(A \cup B) < \delta(A) + \delta(B) - \delta(C)$, which, after rearranging the inequality, contradicts the choice of C. We have a similar contradiction if $\delta(\mathrm{cl}^{\leq}(\bar{a}B)) < \delta(A \cup B)$, thus $A \cup B \leq \mathcal{M}'$. $\square_{Claim}$

So $\mathrm{tp}(\bar{a}/B)$ is determined by C and the isomorphism type of $\mathrm{cl}(\bar{a}C)$. So the number of 1-types over B is at most $max(\aleph_0, |B|)$. Thus T is λ-stable for all infinite λ.

Remarks 1.11. If $A \leq \mathcal{M}' \models T$ is finite then $\mathrm{acl}(A) = \mathrm{cl}^{\leq}(A)$. Indeed, as $\mathrm{cl}^{\leq}(A)$ is finite, we have $\supseteq$. On the other hand if $b \in \mathcal{M}' \setminus \mathrm{cl}^{\leq}(A)$, let $B' = \mathrm{cl}^{\leq}(bA)$ and $A' = \mathrm{cl}^{\leq}(A)$. We can assume that $\mathcal{M}'$ is ω-saturated, so $(C2)$ holds in $\mathcal{M}'$. By considering the free amalgam of copies of B' over A' we obtain infinitely many elements of $\mathcal{M}'$ with the same type as b over A'. It follows that cl is equal to algebraic closure in models of T.

1.6. Forking in $\mathcal{M}$. We describe forking in models of $T = Th(\mathcal{M})$.

Recall the following characterization of forking in a stable (or simple) theory, working in a sufficiently saturated model. Suppose C is a set of parameters and a, b are tuples. Say that a is independent from b over C, written

$$a \mathop{\smile\!\!\!\!\!\mid}_{C} b$$

if:

For every C-indiscernible sequence $(b_i : i < \omega)$ with $\mathrm{tp}(b_i/C) = \mathrm{tp}(b/C)$ there exists a' with $\mathrm{tp}(a'b_i/C) = \mathrm{tp}(ab/C)$ for all $i < \omega$.

One also says that $\mathrm{tp}(a/Cb)$ does not fork over C, or that $\mathrm{tp}(a/Cb)$ is a non-forking extension of $\mathrm{tp}(a/C)$.

Remarks 1.12. What we have defined is non-dividing, which is the same as non-forking in a stable (or simple) theory.

Theorem 1.13. *If $A, B, C \subseteq \mathcal{M}' \models T$ then $A \mathop{\smile\!\!\!\!\!\mid}_C B$ iff*

- $\mathrm{cl}^{\leq}(AC) \cap \mathrm{cl}^{\leq}(BC) = \mathrm{cl}^{\leq}(C)$
- $\mathrm{cl}^{\leq}(AC)$ *and* $\mathrm{cl}^{\leq}(BC)$ *are freely amalgamated over* $\mathrm{cl}^{\leq}(C)$
- $\mathrm{cl}^{\leq}(ABC) = \mathrm{cl} \leq (AC) \cup \mathrm{cl}^{\leq}(BC)$.

Expressed in a different way, if $\bar{a}$ is a tuple in $\mathcal{M}'$, then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_C B$ iff $d(\bar{a}/C) = d(\bar{a}/B)$ and $\mathrm{acl}(\bar{a}C) \cap \mathrm{acl}(B) = \mathrm{acl}(C)$.

Sketch of Proof. Assuming the 3 conditions hold. To simplify the notation we can assume that A, B are closed and have intersection C and we can assume that $\mathcal{M}'$ is highly saturated. We show that $\mathrm{tp}(A/B)$ does not divide over C. Suppose $(B_i : i < \omega)$ is a sequence of translates of B over C. Let X be the $\leq$-closure of the union of these and let Y be the free amalgam of X and A over C. As $B_i \leq X$ we have that A and B_i are freely amalgamated over C and $A \cup B_i \leq Y$. We may assume that $Y \leq \mathcal{M}'$. If A' denotes the copy of A in Y then $\mathrm{tp}(A'B_i) = \mathrm{tp}(AB)$ for each i.

For the converse, we can use the fact that algebraic closure in M_1 is self-sufficient closure to obtain the first bullet point if $A \mathop{\smile\!\!\!\!\!\mid}_C B$. Moreover, we can assume as before that A, B are closed and have intersection C. To simplify the argument, assume also that A, B are finite. Let $(B_i : i < \omega)$ be a sequence of translates of B over A which are freely amalgamated over C and such that the union of any subcollection of them is self-sufficient in M_1. Suppose for a contradiction that A, B are not freely amalgamated over C. Then the same is true of A and B_i and there is $s > 0$ such that $\delta(A \cup B_i) = \delta(A) + \delta(B_i) - \delta(C) - s$ for all i. Then one computes that

$$\delta(A \cup \bigcup_{i=1}^{r} B_i) \leq \delta(\bigcup_{i=1}^{r} B_i) + \delta(A) - C - rs.$$

If r is large enough, this contradicts $\bigcup_{i=1}^{r} B_i \leq M_i$. The third bullet point is similar. $\square$

Remark 1.14. Suppose $B \leq C \leq \mathcal{M}$ and a is a tuple in $\mathcal{M}$. It follows from the above and $\leq$-homogeneity that there is a unique non-forking extension of $\mathrm{tp}(a/B)$ to a type over C. The type over C gives the type of $A' = \mathrm{cl}^{\leq}(aB)$ as specified by $\mathrm{tp}(a/B)$ and says that A', C are freely amalgamated over B and $A' \cup C \leq \mathcal{M}$ (a $\bigwedge$-definable condition).

So types of tuples over $\leq$-subsets of $\mathcal{M}$ are stationary and it follows that if $e \in \mathcal{M}^{eq}$ is algebraic over $B \leq \mathcal{M}$, then it is definable over it.

1.7. Weak elimination of imaginaries. Recall that a structure $\mathcal{N}$ has *weak elimination of imaginaries* if for every $e \in \mathcal{N}^{eq}$ we have $e \in \mathrm{dcl}(\mathrm{acl}(e) \cap \mathcal{N})$.

Theorem 1.15. *The theory $T = Th(\mathcal{M})$ has weak elimination of imaginaries.*

The proof of this is taken from ([**19**], 4.1 and 4.2) where it is attributed to Frank Wagner.

Lemma 1.16. *Suppose $A, B_1, B_2, B \leq \mathcal{M}$ are finite with $B_i \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!\mid}_{B_i} B$. Then $A \mathop{\smile\!\!\!\!\!\!\mid}_{B_1 \cap B_2} B$.*

Proof. Let $A_i = \mathrm{cl}^{\leq}(A \cup B_i)$. So

- $A_i \cap B = B_i$;
- $A_i \cup B \leq \mathcal{M}$;
- A_i, B are freely amalgamated over B_i.

Let $A' = A_1 \cap A_2$ and note that $A' \supseteq A, B_1 \cap B_2$. From the first of the above we obtain $A' \cap B = B_1 \cap B_2$. From the second, by intersecting we obtain $A' \cup B \leq \mathcal{M}$. Finally one checks that A' and B are freely amalgamated over $B_1 \cap B_2$. □

Proof of Theorem: Let $e \in M^{eq}$. So there is a $\emptyset$-definable equivalence relation $E(x, y)$ on some $\emptyset$-definable set of tuples from $\mathcal{M}$ and a tuple a such that e is the E-class a_E. By taking non-forking extensions of $\mathrm{tp}(a/e)$ we can find tuples b_1, b_2 with a, b_1, b_2 independent over e and $a_E = (b_1)_E = (b_2)_E$. As $e \in \mathrm{acl}^{eq}(b_i)$ we have $a \mathop{\smile\!\!\!\!\!\!\mid}_{b_i} b_1 b_2$.

Let $B = \mathrm{acl}(b_1) \cap \mathrm{acl}(b_2)$ (where the acl is in $\mathcal{M}$, so equal to $\mathrm{cl}^{\leq}$). By the lemma, we have $a \mathop{\smile\!\!\!\!\!\!\mid}_B b_1 b_2$. As $e \in \mathrm{acl}(a) \cap \mathrm{acl}(b_1)$ we have $e \in \mathrm{acl}^{eq}(B)$, so by the previous remark, $e \in \mathrm{dcl}^{eq}(B)$.

But also $b_1 \mathop{\smile\!\!\!\!\!\!\mid}_e b_2$ so $B \mathop{\smile\!\!\!\!\!\!\mid}_e B$ whence $B \subseteq \mathrm{acl}^{eq}(e)$. □

2. Complexity of forking

We work throughout with a complete stable L-theory T; much of this can be adapted to simple theories. Unless stated otherwise, all elements will be from a monster model and in general we will not distinguish between real elements and imaginaries.

2.1. One-basedness.

Definition 2.1. We say that T is *one-based* if for every stationary type $\mathrm{tp}(b/A)$ we have $\mathrm{Cb}(b/A) \subseteq \mathrm{acl}(b)$. Equivalently, for all algebraically closed sets A, B we have $A \mathop{\smile\!\!\!\!\!\!\mid}_{A \cap B} B$.

The theory of a pure set, or of a vector space are examples of one-based theories. In general any module is one-based and a theorem of Hrushovski and Pillay analyses the structure of groups in one-based theories, in particular showing that they are abelian by finite. One-basedness is preserved by adding or forgetting parameters.

The notion of a pseudoplane goes back to Lachlan and Zilber in the 1970's; the idea is that it represents an 'incidence structure' of points and curves.

Definition 2.2. We say that a stationary type $p = \mathrm{tp}(bc/A)$ is a *complete type-definable pseudoplane* (in T) if

- $b \notin \mathrm{acl}(cA)$, $c \notin \mathrm{acl}(bA)$
- whenever $b_1c, b_2c \models p$ with $b_1 \neq b_2$ we have $c \in \mathrm{acl}(b_1b_2A)$
- whenever $bc_1, bc_2 \models p$ with $c_1 \neq c_2$ we have $b \in \mathrm{acl}(c_1c_2A)$.

Theorem 2.3. *(Pillay) T is one-based iff there is no complete type-definable pseudoplane in T.*

Examples 2.4. 1. (The free pseudoplane) Take L to have a single binary relation symbol R and unary predicates P_1, P_2. Let T be the theory of an infinitely-branching (unrooted) tree considered as a bipartite graph with the parts labelled by P_1, P_2 and the edges given by R. This is ω-stable of Morley rank ω and has weak elimination of imaginaries. If bc is an adjacent pair of vertices, then $\mathrm{tp}(bc/\emptyset)$ is a complete type-definable pseudoplane in T.

Note that this can be seen as a degenerate case of the Hrushovski construction. In the notation of Section 1.1, take T' to be the theory of (labelled bipartite) graphs without cycles and the predimension $\delta(B) = |B| - |R^B|$. Then $\mathrm{cl}^{\leq}$ is 'convex closure' and the dimension counts the number of connected components in this.

2. Take the Hrushovski construction with graphs and predimension $\delta(B) = 2|B| - |R^B|$; let $abc \leq \mathcal{M}$ with $R(ac), R(bc)$ holding. Then $ab \leq \mathcal{M}$ and $c \leq \mathcal{M}$ and $\mathrm{Cb}(c/ab) = ab \not\subseteq \mathrm{acl}(c)$. So $Th(\mathcal{M})$ is not one-based.

2.2. CM-triviality.

The following terminology is due to Hrushovski.

Definition 2.5. The stable theory T is *CM-trivial* if whenever $A \subseteq B$ are algebraically closed and c is such that $\mathrm{acl}(cA) \cap B = A$, then $\mathrm{Cb}(c/A) \subseteq \mathrm{acl}(\mathrm{Cb}(c/B))$.

Lemma 2.6. *T is CM-trivial iff whenever A, B, C are algebraically closed and $A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} B$, then $A \cap C \mathop{\smile\!\!\!\!\!\!|}_{A\cap B\cap C} B \cap C$.*

Proof. Suppose the condition holds and A, B, c are as in the definition. Let $E = \mathrm{acl}(\mathrm{Cb}(c/B))$ and $F = \mathrm{acl}(cE)$. So $F \mathop{\smile\!\!\!\!\!\!|}_E B$. Intersect with $C = \mathrm{acl}(cA)$ and apply the condition. We obtain $c \mathop{\smile\!\!\!\!\!\!|}_{E\cap A} A$ so $\mathrm{Cb}(c/A) \subseteq E$ as required. The converse is an exercise. □

Example 2.7. Each Hrushovski structure $\mathcal{M}$ as described in Section 1 is CM-trivial. As $\mathcal{M}$ has weak elimination of imaginaries it suffices to check the condition in the above lemma for algebraically closed $A, B, C \subseteq \mathcal{M}$ with $A \mathop{\smile\!\!\!\!\!\!|}_{A\cap B} B$. We can do this using the description of forking in Theorem 1.13. From this we know that $A \cup B \leq \mathcal{M}$, so $(A \cap C) \cup (B \cap C) \leq \mathcal{M}$; also as A, B are freely amalgamated over $A \cap B$, we have that $A \cap C$ and $B \cap C$ are freely amalgamated over their intersection. It then follows from Theorem 1.13 that $A \cap C \mathop{\smile\!\!\!\!\!\!|}_{A\cap B\cap C} B \cap C$.

Example 2.8. An algebraically closed field K is not CM-trivial. Let $a, b, c, d, e, f \in K$ be algebraically independent transcendentals. Consider the following definable sets in K^3:

- P is the plane $\{(x, y, z) \in K^3 : z = ax + by + c\}$;

- ℓ is $\{(x,y,z) \in P : y = dx + e\}$, a line in P;
- p is the point $(f,g,h) \in \ell$ (so $h = af + bg + c$ and $g = df + e$).

As a definable set, P has canonical parameter $[P] = (a,b,c)$. Also, $[\ell] = (d, e, (a+bd), be+c)$ as it is specified by the equations $y = dx + e$ and $z = ax + b(dx+e) + c$. Thinking of transcendence rank, we have $p \mathop{\smile\!\!\!\!\!\!\mid}_{[\ell]} [P]$. So $\mathrm{acl}([\ell]) = \mathrm{acl}(\mathrm{Cb}(p/[\ell], [P])$. We also have $\mathrm{Cb}(p/[P]) = [P]$. So to see non-CM-triviality it will be enough to show:

- $[P] \not\subseteq \mathrm{acl}([\ell])$ - that is, $\{a,b,c\} \not\subseteq \mathrm{acl}(d, e, (a+bd), be+c)$;
- $\mathrm{acl}(p, [P]) \cap \mathrm{acl}([P], [\ell]) = \mathrm{acl}([P])$.

The first follows by rank considerations. The second amounts to showing

$$\mathrm{acl}(f, df + e, a, b, c) \cap \mathrm{acl}(a,b,c,d,e) = \mathrm{acl}(a,b,c),$$

which is an exercise.

Remarks 2.9. CM-triviality is preserved under adding or forgetting constants and passing to imaginary sorts. As far as I know, there is no precise characterisation of CM-trivility in terms of omitting some type of incidence structure. In the finite rank context, a group definable in a CM-trivial structure is nilpotent-by-finite. Baudisch's group is a non-abelian example of such a group not interpretable in an algebraically closed field.

2.3. Ampleness. The following is a slight modification of the definition introduced by Pillay in [**14**].

Definition 2.10. Suppose $n \geq 1$ is a natural number. A complete stable theory T is *n-ample* if (in some model of T, possibly after naming some parameters) there exist tuples $a_0, \ldots, a_n$ such that:
(i) $a_n \not\mathop{\smile\!\!\!\!\!\!\mid} a_0$;
(ii) $a_n \ldots a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_{a_i} a_0 \ldots a_{i-1}$ for $1 \leq i < n$;
(iii) $\mathrm{acl}(a_0) \cap \mathrm{acl}(a_1) = \mathrm{acl}(\emptyset)$;
(iv) $\mathrm{acl}(a_0 \ldots a_{i-1} a_i) \cap \mathrm{acl}(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{acl}(a_0 \ldots a_{i-1})$ for $1 \leq i < n$.
Here acl is algebraic closure in the T^{eq} sense.

Lemma 2.11. (1) *If T is n-ample then it is $n-1$-ample.*
(2) *T is not 1-ample iff it is one-based.*
(3) *T is not 2-ample iff it is CM-trivial.*

Proof. (1) Clear.

(2) Compare with Definition 2.1.

(3) Suppose a_0, a_1, a_2 withness 2-ampleness. We show $c = a_2$, $A = \mathrm{acl}(a_0)$, $B = \mathrm{acl}(a_0, a_1)$ witness non-CM-triviality. Note that $\mathrm{acl}(cA) \cap B = A$. As $a_0 \mathop{\smile\!\!\!\!\!\!\mid}_{a_1} a_2$ we have $\mathrm{Cb}(c/B) \subseteq \mathrm{acl}(a_1)$. So if $\mathrm{Cb}(c/A) \subseteq \mathrm{Cb}(c/B)$ we would have $\mathrm{Cb}(c/A) \subseteq \mathrm{acl}(a_1) \cap A = \mathrm{acl}(\emptyset)$, so $a_0 \mathop{\smile\!\!\!\!\!\!\mid} a_2$ - a contradiction. The converse is similar (or can be obtained using Lemma 2.6). $\square$

Remarks 2.12. It is fairly clear that n-ampleness is preserved under adding or forgetting parameters. A stable theory which (type) interprets an infinite field is n-ample for all n: this is shown in [**14**] following the idea of Example 2.8. Pillay conjectures that there is a non-abelian simple group of finite Morley rank which is not 3-ample (and so is a counterexample to the Cherlin - Zilber Algebraicity Conjecture). The intuition behind this seems to be that the proof that a bad group is 2-ample in [**12**] breaks down for 3-ampleness.

A non-abelian free group is n-ample for all n (a result of Ould Houcine and Tent [**11**]); as far as I'm aware, it is still an open problem whether an infinite field is interpretable.

Recent work of Tent and Baudisch, Martin-Pizarro and Ziegler ([**17, 3**]) shows that Pillay's hierarchy is strict: for every n there is a stable theory which is n-ample but not $n+1$-ample.

2.4. Triviality. The basic reference here is [**7**].

Definition 2.13. A stable theory is *trivial* if, for every three tuples a, b, c of elements and any set A of parameters from some model, if a, b, c are pairwise independent over A, then a, b, c are independent over A.

This is not particularly interesting in the finite rank context: a superstable trivial theory with all types having finite U-rank is one-based ([**7**], Proposition 9). It can be shown that triviality is preserved by adding or forgetting constants and including imaginary sorts. No infinite group is interpretable in a trivial, stable theory (think of the generic type).

Example 2.14. Let T be the theory of the free pseudoplane in Example 2.4 (1). We show that T is trivial. Recall that algebraic closure is convex-closure and non-forking is as given in the Hrushovski examples (Theorem 1.13). Suppose A, B, C are algebraically closed sets which are pairwise independent over (and contain) the algebraically closed set E. So $A \cup B, A \cup B$ and $B \cup C$ are algebraically closed. In general in this example, the algebraic closure of a set is the union of the algebraic closures of its 2-element subsets, and it follows that $A \cup B \cup C$ is algebraically closed. As A, B, C are pairwise freely amalgamated over E, it follows that $A, B \cup C$ are freely amalgamated over E, so $A \mathop{\smile\!\!\!\!\!\!|}_E B \cup C$, as required.

Example 2.15. (Example 2.4 (2) again.) Take the Hrushovski construction with graphs and predimension $\delta(B) = 2|B| - |R^B|$; let $a_1a_2a_3b \leq \mathcal{M}$ with $R(a_ib)$ holding (and no other edges). Then $a_ia_j \leq M$ and so the a_i are pairwise independent. But $a_1 \not\mathop{\smile\!\!\!\!\!\!|} a_2a_3$ as $b \in \mathrm{cl}(a_1a_2a_3)$. So this example is not trivial.

2.5. Reducts. If $\mathcal{M}$, $\mathcal{N}$ are two structures on the same underlying set we say that $\mathcal{M}$ is a *reduct* of $\mathcal{N}$ if (for every n) every $\emptyset$-definable subset of $\mathcal{M}^n$ is also $\emptyset$-definable in $\mathcal{N}^n$. Note that in this case, if $\mathcal{N}_1$ is a model of $Th(\mathcal{N})$ then there is, canonically, a reduct of $\mathcal{N}_1$ which is a model of $Th(\mathcal{M})$. So we also say that $Th(\mathcal{M})$ is a reduct of $Th(\mathcal{N})$. Note also that if we assume that $\mathcal{N}$, $\mathcal{M}$ are Morleyized (so in each case the language has a relation symbol for each $\emptyset$-definable subset), then this accords with the traditional notion of a reduct where one restricts to a sublanguage.

Some properties are preserved under reducts:

Lemma 2.16. *Suppose $\mathcal{M}$ is a reduct of $\mathcal{N}$.*

1. *If $\mathcal{N}$ is λ-stable, so is $\mathcal{M}$.*
2. *If $\mathcal{N}$ is κ-saturated, so is $\mathcal{M}$.*
3. *Suppose $\mathcal{N}$ is stable and saturated. Let $A \subseteq \mathcal{N}$ be small and algebraically closed in $\mathcal{N}$. If $\mathrm{tp}_{\mathcal{N}}(a/Ab)$ does not fork over A (in the sense of $\mathcal{N}$), then $\mathrm{tp}_{\mathcal{M}}(a/Ab)$ does not fork over A (in the sense of $\mathcal{M}$).*

One might expect that forking becomes less complicated when passing to a reduct, but this is not the case.

Example 2.17. (Compare the free pseudoplane, Example 2.4) Take L to have a single binary relation symbol R' and let T' be the theory of an infinitely-branching (unrooted) directed tree tree where each vertex has exactly one directed edge coming out from it. The models of T' consist of disjoint copies of such directed trees; T' is ω-stable of Morley rank ω and has weak elimination of imaginaries. If $X \subseteq M \models T'$ then $\mathrm{acl}(X)$ consists of 'descendants' of the vertices in X, that is, the closure under taking successors. If A, B are algebraically closed then $A \mathop{\smile\!\!\!\!\!\mid}_{A\cap B} B$, so T' is one-based. Moreover, if A, B, C are algebraically closed and pairwise independent over (algebraically closed) E then $B \cup C$ is algebraically closed and $E \subseteq A \cap (B \cup C)$. So $A \mathop{\smile\!\!\!\!\!\mid}_E B \cup C$. It follows that T' is trivial.

Note that we can regard the free pseudoplane as a reduct of this by 'forgetting' the direction on edges. More formally, we take the reduct to the definable relation $R(x, y)$ given by $R'(x, y) \vee R'(y, x)$. This gives an example of a (trivial) one-based theory T' with a reduct T which is not one-based (but which is still trivial). This cannot happen in the finite rank case.

Theorem 2.18. *([**4**]) Suppose T' is a superstable theory in which all types have finite U-rank. If T' is one-based, then every reduct of T' is also one-based.*

There is a corresponding result for CM-triviality.

Theorem 2.19. *([**10**]) Suppose T' is a superstable theory in which all types have finite U-rank. If T' is CM-trivial, then every reduct of T' is also CM-trivial.*

Question 2.20. Is there a corresponding result for 'not n-ample' for $n \geq 2$?

The following (essentailly from [**5, 6**]) shows that triviality is not necessarily preserved under taking reducts, answering a question in [**7**].

Example 2.21. (2-out digraphs: compare the directed version of the free pseudoplane, Example 2.17.) The language L has a single binary relation symbol R' and the theory T'_0 has as its models the class $\bar{\mathcal{D}}_0$ of loopless digraphs in which each vertex has at most two successors, that is, at most two directed edges coming out from it. If $A \subseteq B \in \bar{\mathcal{D}}_0$ write $A \sqsubseteq B$ to mean that it is closed under successors, i.e. if $a \in A$, $b \in B$ and $B \models R'(a, b)$, then $b \in A$. Note that if $X \subseteq B$ there is a smallest successor-closed subset $\mathrm{cl}'_B(X)$ containing it.

Clearly the relation $\sqsubseteq$ is transitive. It also satisfies the following strong version of free amalgamation:

Lemma 2.22. *Suppose $A \sqsubseteq B \in \bar{\mathcal{D}}_0$ and $A \subseteq C \in \bar{\mathcal{D}}_0$. Let F be the free amalgam of B and C over A. Then $C \sqsubseteq F \in \bar{\mathcal{D}}_0$.*

The class $(\mathcal{D}_0; \sqsubseteq)$ is therefore an amalgamation class and so, as in Theorem 1.9, there is a countable generic structure $\mathcal{N} \in \bar{\mathcal{D}}_0$ characterised by the properties:

($D1$): $\mathcal{N}$ is the union of a chain of finite substructures $B_1 \sqsubseteq B_2 \sqsubseteq B_3 \sqsubseteq \ldots$ all of which are in $\mathcal{D}_0$.

($D2$): If $A \sqsubseteq \mathcal{N}$ is finite and $A \sqsubseteq B \in \mathcal{D}_0$, then there is an embedding $f : B \to \mathcal{N}$ with $f(B) \sqsubseteq \mathcal{N}$ and which is the identity on A.

Theorem 2.23. *Let $\mathcal{M}$ denote the undirected reduct of the digraph $\mathcal{N}$. Then $\mathcal{M}$ is the Hrushovski structure constructed from the predimension $\delta(A) = 2|A| - |R^A|$ as in Theorem 1.9. In particular, $\mathcal{M}$ is neither trivial nor one-based.*

Proof. We sketch the main ideas. The key point is the following lemma, the first part of which is a consequence of Hall's Marriage Theorem.

Lemma 2.24. *Let $\mathcal{C}_0$ denote the class of finite graphs A with $\emptyset \leq A$, as in Section 1.1, using the given predimension. Then:*

(1) *$\mathcal{C}_0$ is precisely the class of undirected reducts of the digraphs in $\mathcal{D}_0$. Moreover, if $A \subseteq B \in \mathcal{C}_0$ then $A \leq B$ iff the edges in B can be directed to give a digraph $B' \in \mathcal{D}_0$ with $A \sqsubseteq B'$.*
(2) *Suppose $A \sqsubseteq B \in \mathcal{D}_0$ and let $A' \in \mathcal{D}_0$ have the same vertex set and undirected reduct as A. Let B' be the result of replacing A by A' in B. Then $A' \sqsubseteq B' \in \mathcal{D}_0$.*

We now show that $\mathcal{M}$ satisfies the two conditions in Theorem 1.9. Condition $(C1)$ follows from $(D1)$ and (1) of the Lemma. For $(C2)$, suppose $A \leq \mathcal{M}$ is finite and $A \sqsubseteq B \in \mathcal{C}_0$. By (1), the edges of B can be directed to give a digraph $B' \in \mathcal{D}_0$ so that $A \sqsubseteq B'$. By (2) we can assume that the induced subdigraph on A is the same as in $\mathcal{N}$. There is a finite $C \sqsubseteq \mathcal{N}$ with $A \subseteq \mathcal{N}$. The free amalgam F of B' and C over A is in $\mathcal{D}_0$ and $C \sqsubseteq F$. Applying $(D2)$ and taking reducts then gives what we want for $(C2)$. This finishes the proof of Theorem 2.23. □

The point of the example is then to observe:

Lemma 2.25. *$Th(\mathcal{N})$ is stable, trivial and one-based.*

So the Hrushovski construction $\mathcal{M}$ - which is neither trivial nor one-based - is a reduct of the trivial, one-based structure $\mathcal{N}$.

Proof. (of Lemma.) Let Σ' consist of axioms describing the class of finite structures $\mathcal{D}_0$, together with the axioms $\sigma_{X,Y}$ for $X \sqsubseteq Y \in \mathcal{D}_0$ stating that for every copy X' of X there is a copy Y' of Y containing it in which the successors of all vertices in $Y' \setminus X'$ are in Y'. One then shows that:

(a) $\mathcal{N} \models \Sigma'$;

(b) if $\mathcal{N}_1, \mathcal{N}_2$ are ω-saturated models of Σ' then the set of isomorphisms $f : A_1 \to A_2$ where $A_i \sqsubseteq \mathcal{N}_i$ are finitely generated, is a back-and-forth system.

So Σ' axiomatises $Th(\mathcal{N})$ and types are determined by quantifier-free types of closures. Note that, in a model of $Th(\mathcal{N})$, for closed A and a tuple b we have that $\mathrm{cl}'(bA)$ is the free amalgam of $\mathrm{cl}'(b)$ and A over their intersection, so one can count types to obtain stability. One easily shows that two closed sets are independent over their intersection and this gives one-basedness; triviality follows as the closure of a set is the union of the closures of its elements.

□

Question 2.26. Is a reduct of finite U-rank of a trivial, one-based stable theory necessarily one-based? Are Hrushovski's strongly minimal structures in [**9**] reducts of trivial theories?

Question 2.27. Is a reduct of a superstable trivial (one-based) theory necessarily trivial?

3. Obtaining ampleness

3.1. The Baudisch-Pillay pseudospace. We shall describe the 'free pseudospace' constructed by Baudisch and Pillay [**2**], a structure which is 2-ample and trivial. This construction has recently been extended by Tent [**17**] and Baudisch, Martin-Pizarro and

Ziegler [**3**] to obtain 'free n-spaces' which are ω-stable, trivial and n-ample, but not $n+1$-ample. Our presentation is a mixture of that in [**2**] and the more recent [**3, 17**].

Definition 3.1. Let L be a first-order language with unary predicates $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ and a binary relation symbol R. A *coloured* 2*-space* is a tripartite (undirected) graph A with vertices $\mathcal{A}_0(A), \mathcal{A}_1(A), \mathcal{A}_2(A)$ referred to, respectively as *points, lines* and *planes*, such that adjacencies (given by R) are between points and line and between lines and planes.We refer to adjacency here as *incidence*.

We could make the corresponding definition of a coloured N-space for any $N \geq 1$. Note that the free pseudoplane, a model of the theory described in Example 2.4 (1), is a coloured 1-space.

Definition 3.2. Suppose A is a coloured 2-space.

(1) If a is a plane in A let $\mathcal{A}_1(a)$ consist of the lines incident with a and $\mathcal{A}_0(a)$ the set of points incident with these (the points *on* the plane a). Similarly if c is a point, define $\mathcal{A}_1(c), \mathcal{A}_2(c)$.
(2) If $d, d' \in A$ a *path* between d and d' is a sequence $d = d_0, d_1, \ldots, d_n = d'$ of adjacent vertices with no repetitions except possibly $d = d'$. If $d = d'$ we refer to this path as a *circle*. The *length* of a path is the number of distinct vertices in it.

Definition 3.3. The set Σ of axioms expresses that we are working with a coloured 2-space and the following hold:

(Σ1) (a): the points and lines form a free pseudoplane; dually, (b) the lines and planes form a free pseudoplane.

(Σ2) (a): for every plane a, $\mathcal{A}_0(a), \mathcal{A}_1(a)$ is a free pseudoplane; (b) the dual.

(Σ3) (a): If a, a' are distinct planes the set of points on both of them is empty, a singleton, or the set of points incident to a line (in $\mathcal{A}_1(a) \cap \mathcal{A}_1(a')$); (b) the dual.

$(\Sigma 4)_n$ (a): Suppose a is a plane and $a, b, \ldots, b', a$ is a circle of length n. Then there is a path between b, b' of length at most $n-1$ which consists only of points and lines incident with a and the points come from the original circle.

$(\Sigma 4)_n$ (b): the dual.

A *free pseudospace* is a model of Σ.

Theorem 3.4. *([**2**]) The L-formulas Σ axiomatise a complete consistent theory which is ω-stable, trivial and* 2*-ample. Moreover, if a_0, a_1, a_2 are an incident point, line, plane triple, then a_0, a_1, a_2 withnesses* 2*-ampleness (over $\emptyset$).*

We will not prove this (it is most of [**2**]). Instead we describe a Fraïssé-style construction of a model of Σ along the lines given in [**3**] and [**17**] (though some aspects are present in [**2**]). In the following, by a *flag* in a coloured 2-space A we mean an incident point-line-plane triple a_0, a_1, a_2.

Definition 3.5. Suppose B is coloured 2-space and $A \subseteq B$. We say that B is obtained from A by a *strong operation* if $A = B$ or one of the following holds:

- $B \setminus A$ is a flag with no incidences to elements of A;
- $B \setminus A$ is a point or plane incident to a single line in A;
- $B \setminus A$ is a line b_1 and there is a flag a_0, a_1, a_2 in A such that a_0, b_1, a_2 is a flag, and there are no other incidences involving b_1;

- $B \setminus A$ is an incident line, plane pair b_1, b_2 such that b_1 is incident to a single point in A and no planes, and b_2 is not incident to anything in A.
- The dual of the above.

We say that $A \leq B$ if B can be obtained from A by a (possibly transfinite) sequence of strong operations $A \leq A_1 \leq A_2 \leq \ldots \leq B$.

We let $\bar{\mathcal{C}}$ denote the class of coloured 2-spaces B with $\emptyset \leq B$; let $\mathcal{C}$ denote the finite ones.

Note that if $B \in \bar{\mathcal{C}}$ then every vertex is in a flag. The relation $\leq$ is clearly transitive, and it is easy to see that if $\emptyset \leq A \leq B_1, B_2 \in \bar{\mathcal{C}}$ then the free amalgam F of B_1 and B_2 over A is in $\bar{\mathcal{C}}$. We then have the following:

Proposition 3.6. *There is a sequence of finite structures (in $\mathcal{C}$)*

$$\emptyset \leq B_1 \leq B_2 \leq B_3 \leq \ldots$$

with the 'richness' property that if $\emptyset \leq A \leq B_i$ and $A \leq C \in \mathcal{C}$, then there is $j \geq i$ and an embedding $f : C \to B_j$ with $f(C) \leq B_j$ and which is the identity on A.

Let $\mathcal{M}_\infty$ denote the union of this sequence. Then the countable structure $\mathcal{M}_\infty$ is uniquely determined up to isomorphism by these properties and is $\leq$-homogeneous (meaning: if $\emptyset \leq A_1, A_2 \leq \mathcal{M}_\infty$ and $f : A_1 \to A_2$ is an isomorphism, then f extends to an automorphism of $\mathcal{M}_\infty$).

Theorem 3.7. *The countable structure $\mathcal{M}_\infty$ is an ω-saturated model of Σ.*

Proof. We sketch the fact that $\mathcal{M}_\infty \models \Sigma$ (the ω-saturation requires a full understanding of the types). The parts of the pseudoplane conditions in $(\Sigma 1, 2)$ which relate to there being infinitely many points on a line etc. follow from the richness property in the definition of $\mathcal{M}_\infty$.

For the remaining parts of $(\Sigma 1, 2)$, which state that there are no cycles in the point-line structure or the line-plane structure, and for the remaining axioms, we check that these properties are preserved under strong operations.

For example, suppose B is obtained from A by a strong operation and $(\Sigma 4)$ holds in A. Suppose a is a plane in B and $C = a, b, \ldots, b', a$ is a circle in B which is not in A. Then B must have been obtained from A by adding a new line b_1 forming a flag a_0, b_1, a_2 with vertices $a_0, a_2 \in A$. By assumption, there is $a_1 \in A$ such that a_0, a_1, a_2 is a flag. The flag a_0, b_1, a_2 must appear in C. We now consider various cases.

Case 1: $b, b' \in A$ and $a_1 \notin C$. Then C' obtained by replacing b_1 in C by a_1 is a circle in A with the same points as C. Apply $(\Sigma 4)$ in A to this.

Case 2: $b, b' \in A$ and $a_1 \in C$. So there is a (shorter) circle in A of the form $ab \ldots a_1 \ldots b'a$ with points a subset of those in C. Apply $(\Sigma 4)$ to this in A.

Case 3: $b = b_1$ (so $a = a_2$) and $a_1 \notin C$. Let C' be the circle in A as in Case 1. Find an appropriate path $a_1 \ldots b$ in A. If this uses a_0 we have a path $b_1 a_0 \ldots b'$ of the same length. If not, it is of length $\leq n - 3$ (where n is the length of C) and so $b_1 a_0 a_1 \ldots b'$ is an appropriate path of length $\leq n - 1$, as required.

Case 4: $b = b_1$ and $a_1 \in C$. If $a_1 = b'$, use $b_1 a_0 a_1$ as the required path. Otherwise we can 'shorten' the part of the circle in A as in Case 2 and apply the argument of Case 3.

□

Theorem 3.7 can be used to show that a flag $a_0a_1a_2 \leq \mathcal{M}_\infty$ witnesses 2-ampleness: the proof is very similar to the proof of Theorem 3.15 given below.

3.2. Ampleness via reducts. We have already seen that some of the constructions witnessing various degrees of ampleness have 'directed' counterparts which are trivial and one-based. Specifically,

- The free pseudoplane (1-ample) is a reduct of the directed free pseodoplane (one-based) with out-valency 1.
- The Hrushovski construction with graphs and a predimension 'twice number of vertices minus number of edges' (1-ample) is a reduct of the generic structure (trivial, one-based) arising from finite 2-out digraphs.

Recall also that the free pseudoplane can also be seen as a degenerate case of the Hrushovski construction.

The construction of $\mathcal{M}_\infty$ via strong operations suggests that there is a directed counterpart where the pseudoplanes involved are of out-valency 1, and it was shown by Grunert [**8**] that $\mathcal{M}_\infty$ is indeed a reduct of a trivial, one-based structure.

These considerations suggested the approach in [**5**] for obtaining a non-trivial stable structure which is n-ample as a reduct of a 'directed' trivial, one-based stable structure (where the out-valencies are finite but greater than 1). Unfortunately there is a gap in the argument in [**5**] (specifically in the proof of the stability of the directed structure in ([**5**], Lemma 2.4)). By modifying the argument, I can fill the gap in the cases $n = 2, 3$. I will finish these notes by describing the construction in the case $n = 2$. The argument for $n = 3$ is significantly harder.

Definition 3.8. Work with the language L of Definition 3.1 and consider the class $\bar{\mathcal{D}}^0$ of *directed* coloured 2-spaces where each vertex has at most 2 directed edges coming out of it. If $A \subseteq D \in \bar{\mathcal{D}}^0$, write $A \sqsubseteq D$ if A is closed under successors (in D) and let cl'_D denote the operation of successor-closure in D.

We let $\bar{\mathcal{D}}$ consist of the structures $A \in \bar{\mathcal{D}}^0$ which satisfy the following additional property: If

$$a_0 \leftarrow a_1 \rightarrow a_2$$

is a flag in A, there is $b_1 \in \mathcal{A}_1(A)$ with

$$a_0 \rightarrow b_1 \rightarrow a_2 \text{ or } a_0 \rightarrow b_1 \leftarrow a_2 \text{ or } a_0 \leftarrow b_1 \leftarrow a_2.$$

We refer to $a_0 \leftarrow a_1 \rightarrow a_2$ here as a *problem path* and a_0, b_1, a_2 as a *solution* to it.

The point of the extra condition is the following:

Lemma 3.9. *Suppose $B \sqsubseteq A \in \bar{\mathcal{D}}$. Then $B \in \bar{\mathcal{D}}$. Moreover, if $b_0, b_2 \in B$ are of types $0, 2$ and there is $a_1 \in A$ of type 1 such that b_0, a_1, b_2 is a flag, then there is a flag b_0, b_1, b_2 in B.*

We have the following amalgamation lemma:

Lemma 3.10. *Suppose $B, C \in \bar{\mathcal{D}}$ and $A \subseteq B$, $A \sqsubseteq C$. Then the free amalgam F of B, C over A is in $\bar{\mathcal{D}}$.*

Remarks 3.11. By using a Fraïssé - style construction, one can show that there is some $N \in \bar{\mathcal{D}}$ with the property that:

(*) If $U \sqsubseteq N$ is finitely generated and $U \sqsubseteq V \in \bar{\mathcal{D}}$ is finitely generated, there is an embedding $f : V \to N$ with $f(V) \sqsubseteq N$ and which is the identity on U.

We refer to such stuctures as *rich*. We would like to produce a consistent theory T_2 containing the axioms Σ for $\bar{\mathcal{D}}$ with the property that all of its sufficiently saturated models are rich. It will then follow that T_2 is complete and the type of a tuple is determined by the quantifier free type of its closure, using a standard back-and-forth argument.

LEMMA 3.12. *Suppose $C \subseteq V \in \bar{\mathcal{D}}$ is closed under successors of vertices of types* 0, 2. *Then $C \in \bar{\mathcal{D}}$. In particular, if $C_0 \subseteq V$ is finite, there is a finite $C \subseteq V$ containing C_0 with $C \in \bar{\mathcal{D}}$ (and $|C|$ can be bounded in terms of the size of C_0).*

PROOF. If $a_0 \leftarrow a_1 \rightarrow a_2$ is a problem path in C then it has a solution a_0, b, a_2 in V. By considering the possible directions on the edges in this, one sees that b must be a successor of a_0 or a_2, so by assumption on C, it is in C.

The second part follows from the first by taking C to be the set of successors in V (necessarily of type 1) of vertices of types $0, 2$ in C_0. The boundedness comes from the bound on the number of successors in V. □

As in 2.1 of [**5**], for each pair $X \sqsubseteq A \in \bar{\mathcal{D}}$ with A finite, there is a closed formula $\sigma_{X,A}$ such that if $V \in \bar{\mathcal{D}}$ then $V \models \sigma_{X,A}$ iff for every embedding $g : X \to V$, there is an extension $f : A \to V$ such that $\mathrm{cl}'(f(A))$ is the free amalgam of $\mathrm{cl}'(X)$ and $f(A)$ over X. (To express the latter condition, it suffies to indicate that the successors of elements of $f(A \setminus X)$ are in $f(A)$.)

Let T_2 consist of the axioms Σ describing $\bar{\mathcal{D}}$, together with all these $\sigma_{X,A}$.

LEMMA 3.13.
(1) *If $N \in \bar{\mathcal{D}}$ is rich, then $N \models T_2$.*
(2) *If $N \models T_2$ is sufficiently saturated, then it is rich.*

PROOF. (1) Suppose $g : X \to N$ is an embedding and let U be the closure of $g(X)$ in N. By the amalgamation lemma 3.10, the free amalgam V of A and U over X is in $\bar{\mathcal{D}}$ (and is finitely generated), and $U \sqsubseteq V$. So by richness, we can extend g in the required way.

(2) Suppose $U \sqsubseteq N$ and $U \sqsubseteq V$ are finitely generated. Given a finite $C_0 \subseteq V$ we can find a finite $C \subseteq V$ containing it which is in $\mathcal{D}$ (Lemma 3.12). We can also ensure that C contains all successors in U of $C \setminus U$. Then $X = U \cap C \sqsubseteq C$ and U, C are freely amalgamated over X. Moreover, if $Y \subseteq U$ is finite and closed under successors of points of types $0, 2$, then (by Lemma 3.12), $Y \in \bar{\mathcal{D}}$ and $Y \cup C$ is the free amalgam of Y and C over X, and is in $\bar{\mathcal{D}}$. So we can use the axiom $\sigma_{Y,Y\cup C}$ in N to obtain a copy of C over Y in N in which all successors of points in $C \setminus X$ are in X.

By increasing Y and using saturation (say, ω_1-saturation, but ω-saturation will suffice as U is finitely generated), we obtain a copy of C over U which is freely amalgamated with U over X and whose union with U is closed in N. This is not quite what we want (because of course, elements of C may have successors in V which are outside U). It does however show consistency of the set of formulas consisting of the basic diagram of U, the basic diagram of V and formulas expressing that the successor of a point in V is again a point of V. So by saturation, there is a $\sqsubseteq$-closed copy of V over U in N, as required for richness. □

COROLLARY 3.14.
(1) *The theory T_2 is complete and consistent and isomorphisms between closed substructures of models of T_2 are elementary. Algebraic closure in a model of T_2 is equal to $\sqsubseteq$-closure.*

(2) *The theory T_2 is stable. If A, B, C are closed subsets of a model of T_2 with $A \cap B = C$, then $A \mathop{\smile\!\!\!\!\!\mid}_C B$. It follows that T_2 is one-based and trivial.*

PROOF. This is all as described for Lemma 2.4 of [**5**] but with the above lemma filling the gap. □

Let $\mathcal{N}$ be a large saturated model of T_2 and $\mathcal{M}$ the undirected reduct of $\mathcal{N}$. Then $\mathcal{M}$ is stable and a saturated model of its theory. In the following, we denote the undirected edge relation in the reducts by r.

THEOREM 3.15. *We have that $Th(\mathcal{M})$ is 2-ample and non-trivial.*

The proof of this is as in [**5**], but we outline the details here.

DEFINITION 3.16. Let $\bar{\mathcal{C}}$ denote the class of undirected reducts of the structures in $\bar{\mathcal{D}}$. If $C \in \bar{\mathcal{D}}$ is the reduct of $D \in \bar{\mathcal{D}}$, refer to D as an *orientation* of C. If $A \subseteq C \in \bar{\mathcal{C}}$ write $A \leq C$ to mean that there is an orientation of C in which A is a succesor-closed subset.

LEMMA 3.17. (1) *If $A \sqsubseteq D \in \bar{\mathcal{D}}$ and $A' \in \bar{\mathcal{D}}$ has the same undirected reduct as A, then the result D' of replacing A in D by A' is also in $\bar{\mathcal{D}}$.*
(2) *The relation $\leq$ is transitive on $\bar{\mathcal{C}}$ and $\bar{\mathcal{C}}$ is closed under free amalgamation over $\leq$-subsets.*

Recall that $\mathcal{N}$ is a large saturated model of T_2 and $\mathcal{M}$ its undirected reduct. In the following, 'small' means of cardinality less than that of $\mathcal{N}$.

PROPOSITION 3.18. (1) *If $A \sqsubseteq \mathcal{N}$ is small and $A' \in \bar{\mathcal{D}}$ has the same undirected reduct as A, then the result $\mathcal{N}'$ of replacing A in $\mathcal{N}$ by A' is a saturated model of T_2.*
(2) *If $A \subseteq \mathcal{M}$ is small then $A \leq \mathcal{M}$ iff there is an orientation of $\mathcal{M}$ which is a saturated model of T_2 in which A is closed.*
(3) *If $A \leq \mathcal{M}$ is small and $\beta : A \to B$ is a $\leq$-embedding with $B \in \bar{\mathcal{C}}$ small, there is a $\leq$-embedding $\gamma : B \to \mathcal{M}$ with $\gamma \circ \beta$ the identity on A.*
(4) *If $A_1, A_2 \leq \mathcal{M}$ are small and $\alpha : A_1 \to A_2$ is an isomorphism then α can be extended to an automorphism of $\mathcal{M}$.*

We do not have a full characterization of forking in $\mathcal{M}$, but the following is useful. The proof is similar to that for the Hrushovski construction given in Theorem 1.13.

PROPOSITION 3.19. *Suppose A, B, C are small $\leq$-subsets of $\mathcal{M}$ with $A \cap B = C$, $A \cup B \leq \mathcal{M}$ and $A \cup B$ the free amalgam of A, B over C. Then $A \mathop{\smile\!\!\!\!\!\mid}_C^{\mathcal{M}} B$.*

We now verify the 2-ampleness in Theorem 3.15.

Let $A = \{a, b, c\} \sqsubseteq \mathcal{N}$ be such that a, b, c is a flag in $\mathcal{N}$. So $abc \leq \mathcal{M}$. We claim:

(i) $a \mathop{\smile\!\!\!\!\!\mid}_b^{\mathcal{M}} c$;
(ii) $a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathcal{M}} c$;
(iii) $\mathrm{acl}(a) \cap \mathrm{acl}(b) = \mathrm{acl}(\emptyset)$;
(iv) $\mathrm{acl}(ab) \cap \mathrm{acl}(ac) = \mathrm{acl}(a)$.

(i) We can orient A as $a \to b \leftarrow c$. Thus $\{b\}, \{a, b\}, \{c, b\} \leq \mathcal{M}$, and so (i) follows from Proposition 3.19.

(ii) As in (i) but using a different orientation we have that $\{c\} \leq \mathcal{M}$. Let $C = \{c_i : i < \omega\} \leq \mathcal{M}$ be of type 2. Thus $c_i \leq \mathcal{M}$ for each i, and these are indiscernible and of the same type as c over $\emptyset$. We show that there is no $a' \in \mathcal{M}$ with $\mathrm{tp}(a'c_i) = \mathrm{tp}(ac)$ for all i. Suppose there is such an a'. In particular, there is a flag $a'b_ic_i$ for each i. There is an orientation of $\mathcal{M}$ in which C is $\sqsubseteq$-closed. We can assume that each $a'b_ic_i$ is not a problem path, so as c_i is closed in the orientation, we must have $a' \to b_i \to c_i$. As there can be at most 4 such directed paths from a' we have a contradiction.

(iii) Suppose $e \in \mathrm{acl}(a) \cap \mathrm{acl}(b)$. There is a sequence $\{b_j : j < \omega\}$ of distinct elements of $\mathcal{M}$ of type 1 with $b = b_0$, $\mathcal{M} \models r(a, b_j)$ for each j, and $B = \{a, b_0, b_1 \ldots\} \leq \mathcal{M}$. Then $ab_j \leq \mathcal{M}$ and the b_j are all of the same type over a (by automorphisms). The same is true of any pair of the b_j. As $e \in \mathrm{acl}(a)$ it follows that b_0, b_1 have the same type over ae.

Thus $e \in \mathrm{acl}(a) \cap \mathrm{acl}(b_1)$, so $e \in \mathrm{acl}(b_0) \cap \mathrm{acl}(b_1)$. Now, Proposition 3.19 shows that $b_0 \mathop{\smile\!\!\!\!\!|}^{\mathcal{M}} b_1$, which implies $\mathrm{acl}(b_0) \cap \mathrm{acl}(b_1) = \mathrm{acl}(\emptyset)$.

(iv) This is similar to (iii). Take $e \in \mathrm{acl}(ab) \cap \mathrm{acl}(ac)$. There exist distinct $(b_j : j < \omega)$ of type 1 with $b = b_0$ and $D = \{a, c, b_j : j < \omega\} \leq \mathcal{M}$ with $r(a, b_j) \wedge r(b_j, c)$ for all j being the only atomic relations on D. To see this, note that the orientation with

$$R(a, b_0) \wedge R(c, b_0) \wedge \bigwedge_{j>0} R(b_j, a) \wedge R(b_j, c)$$

is in $\mathcal{D}$. By replacing b_0 by any of the other b_j, we see that $ab_jc \leq D \leq \mathcal{M}$. In particular, the b_j are of the same type over ac. As $e \in \mathrm{acl}(ac)$ we can therefore assume that b_0, b_1 are of the same type over ace, so $e \in \mathrm{acl}(ab_0) \cap \mathrm{acl}(ab_1)$. By choosing a different orientation we can see $ab_0 \leq ab_0b_1 \leq ab_0b_1c \leq \mathcal{M}$ (– take an orientation of D where b_0, b_1 are successors of a and successors of c). Thus by Proposition 3.19 again, $b_0 \mathop{\smile\!\!\!\!\!|}_a^{\mathcal{M}} b_1$ and so $e \in \mathrm{acl}(a)$.

Question 3.20. Is $Th(\mathcal{M})$ superstable?

I suspect that the answer is 'no', but do not have a proof.

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School of Mathematics, University of East Anglia, Norwich NR4 7TJ, England

E-mail address: d.evans@uea.ac.uk