Adding Skolem functions to simple theories

Herwig Nübling

13th October 2003

Abstract

We examine the conditions under which we can keep simplicity or categoricity after adding a Skolem function to the theory.
AMS classification: 03C45 , 03C50
Keywords: Model theory; Skolem function; Simple

Introduction

The motivation for this paper lies in finding a way of turning algebraic closure into definable closure in a first order theory T by means of adding Skolem functions to T, but without losing properties such as simplicity or categoricity. In particular any strongly minimal structure (respectively any ω-categorical SU-rank 1 structure) can be expanded to a structure which is SU-rank 1 (respectively, and ω-categorical) and in which algebraic closure equals definable closure. One could further note one of Zilber's theorems stating that a strongly minimal structure in which $\mathrm{acl} = \mathrm{dcl}$ is locally modular (see Theorem 5.6. in [5]).
Recall that a theory T is *algebraically bounded* if for every formula $\phi(x, \bar{y})$ there is an integer n_ϕ such that for every $M \models T$ and $\bar{b}$ from M, the set $\{a \in M \mid M \models \phi(a, \bar{b})\}$ is either infinite or of size $\leq n_\phi$. Recall further that T is algebraically bounded if and only if T *eliminates the quantifier* $\exists^\infty$, i.e. for every formula $\phi(x, \bar{y})$ and every $M \models T$ the set of tuples $\bar{b}$ from M such that $\{a \in M \mid M \models \phi(a, \bar{b})\}$ is infinite is definable. Winkler showed in [4] that if T is a model-complete, algebraically bounded theory then any Skolem expansion of T has a model-completion. This is the theorem that provided the starting point for this paper.
In the first chapter we show that if we add a Skolem function for an algebraic formula to an algebraically bounded, model-complete, simple theory, then its model companion is still simple. The proof follows the proofs in [2], which showed that

if we add a new, unary predicate to a simple theory then its model companion is simple. When adding Skolem functions for all algebraic formulas we get a simple theory in which algebraic closure and definable closure coincide.
The second chapter shows a way of adding Skolem functions to an ω-categorical theory such that its model companion is also ω-categorical. We show through the example of algebraically closed fields that we can normally not keep uncountable categoricity.
In the third chapter it is demonstrated that if we add the Skolem function for an arbitrary formula in a simple theory, its model companion will not be simple in general.
The fourth chapter concerns adding a unary, generic function f to a theory T which has quantifier elimination and in which the algebraic closure of a set is equal to the set itself, to obtain the theory T_f. If T is stable or simple then also T_f is stable or simple. In particular that shows that if we add to such a simple theory T the Skolem function for a nonalgebraic formula like '$x \neq y$', its model companion is simple.
The author would like to thank his supervisor David Evans for providing him with the idea for this paper, as well as for his constant encouragement. The author would also like to thank the referee for the helpful comments.

1 Adding an algebraic Skolem function

Let T be a complete, model-complete, algebraically bounded theory in a language L. We are working in a big model $\mathbf{C}$.

Recall that if $\psi(\bar{y}, x)$ is an L-formula then we can add a new function symbol f , the *Skolem function* for the formula $\psi(\bar{y}, x)$, to L and we obtain in this new language L^+ a *Skolem expansion* T^+ of T such that

$$T^+ \quad = \quad T \cup \{\forall\, \bar{y}\ (\exists\, x\, \psi(\bar{y}, x) \to \psi(\bar{y}, f(\bar{y})))\}$$

Definition 1.1 Let $\psi(\bar{y}, x)$ be an L-formula, $\bar{y} = y_0, y_1, \ldots y_k$ and n_ψ an algebraic bound for $\psi(\bar{y}, x)$ in x. By *adding the algebraic Skolem function* $f(\bar{y})$ *for the formula* $\psi(\bar{y}, x)$ we mean adding the Skolem function for the formula

$$(\psi(\bar{y}, x) \wedge \neg\exists^{>n_\psi} v\, \psi(\bar{y}, v)) \vee ((\neg\exists v\, \psi(\bar{y}, v) \vee \exists^{>n_\psi} v\, \psi(\bar{y}, v)) \wedge x = y_0)\,.$$

If $\psi(x)$ is a formula in one variable we can add the algebraic Skolem function, which is constant, only if there is $0 < n < \omega$ such that

$$\mathbf{C} \models \exists^{=n} x\, \psi(x).$$

Let now T^+ be the Skolem expansion of T after adding the algebraic Skolem function $f(\bar{y})$ for a formula $\psi(\bar{y}, x)$. Let T^* be the model-completion of T^+ which we know exists by Theorem 2 of [4].

Let $\mathbf{C}^*$ be an expansion of $\mathbf{C}$ to a big model of T^*. If X is a small subset of $\mathbf{C}^*$ we denote by $\mathrm{acl}^*_T(X)$ the algebraic closure of X in the sense of T^* and by $\mathrm{acl}_T(X)$ the algebraic closure of X in the sense of T. Similarly we denote for $a \in \mathbf{C}^*$ and X a small subset of $\mathbf{C}^*$ by $\mathrm{tp}_T(a/X)$ the type of a over X in the sense of T.

Note that for all $\bar{a} \in \mathbf{C}^*$ we have $f(\bar{a}) \in \mathrm{acl}_T(\bar{a})$.
In the following lemmas, which are similar to the ones in [2], we are going to show that for all sets A in a model of T^* we have $\mathrm{acl}_T(A) = \mathrm{acl}_{T^*}(A)$.

Lemma 1.2 *Let (M_0, f_0) and (M_1, f_1) be models of T^* and A a common subset of M_0 and M_1. Then*

$$(M_0, f_0) \equiv_A (M_1, f_1) \Leftrightarrow \left(\mathrm{acl}_T(A), f_0|_{\mathrm{acl}_T(A)}\right) \simeq_A \left(\mathrm{acl}_T(A), f_1|_{\mathrm{acl}_T(A)}\right)$$

Proof: Left to right is standard.
Assume now that the right hand side of the equivalence holds. We may assume that $A = \mathrm{acl}_T(A)$ and that $M_0 \cap M_1 = A$, since otherwise we can find a model (M_2, f_2) of T^* such that $M_2 \cap M_0 = M_2 \cap M_1 = A$ and $\left(\mathrm{acl}_T(A), f_0|_{\mathrm{acl}_T(A)}\right) \simeq_A \left(\mathrm{acl}_T(A), f_2|_{\mathrm{acl}_T(A)}\right)$. We may also assume that $f_0|_{\mathrm{acl}_T(A)} = f_1|_{\mathrm{acl}_T(A)}$ since for all $\bar{a} \in A$ also $f_i(\bar{a}) \in A$. Let N be a model of T containing M_0 and M_1. Define f on N such that $f|_{M_0} = f_0$ and $f|_{M_1} = f_1$ and $(N, f) \models T^+$. Hence (N, f) embeds in a model (N', f') of T^*. But then $(M_0, f_0) \preceq (N', f')$ and $(M_1, f_1) \preceq (N', f')$ and so $(M_0, f_0) \equiv_A (M_1, f_1)$. $\Box$*Lemma*

Lemma 1.3 *The completions of T^* are arrived at by describing $f|_{acl_T(\emptyset)}$.*

Proof: If (M_0, f_0) and (M_1, f_1) are models of T^* such that $f_0|_{acl_T(\emptyset)} \simeq f_1|_{acl_T(\emptyset)}$ then $(M_0, f_0) \equiv (M_1, f_1)$ by Lemma 1.2. $\Box$*Lemma*

Lemma 1.4 *If $M \models T^*$, $A \subseteq M$, and $\bar{a}, \bar{b} \in M$ then $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ iff there is an A-isomorphism between $\mathrm{acl}_T(A, \bar{a})$ and $\mathrm{acl}_T(A, \bar{b})$ which takes $\bar{a}$ to $\bar{b}$ and preserves f.*

Proof: If $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$ then we can find such an isomorpism.
Let now g be the f-preserving A-isomorphism between $\mathrm{acl}_T(A, \bar{a})$ and $\mathrm{acl}_T(A, \bar{b})$

such that $g(\bar{a}) = \bar{b}$. Let N be a model of T^* which contains A and $\bar{a}$. Extend g to N and let $g(N)$ be the image of N under g. Then $f^{g(N)}|_{acl_T(A,\bar{b})} = f^M|_{acl_T(A,\bar{b})}$. So by Lemma 1.2 $g(N) \equiv_{A,\bar{b}} M$. So $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$. $\square$*Lemma*

Lemma 1.5 *Let $M^* \models T^*$, $a \in M^*$ and $A \subseteq M^*$. Then a is algebraic over A iff $a \in \mathrm{acl}_T(A)$. Hence $\mathrm{acl}_T(A) = \mathrm{acl}_{T^*}(A)$.*

Proof: Assume that $c \notin \mathrm{acl}_T(A)$. Let $\hat{c}$ be a realisation of $\mathrm{tp}_T(c/\mathrm{acl}_T(A))$ such that $\mathrm{acl}_T(A, \hat{c}) \cap M = \mathrm{acl}_T(A)$. Let N be a model of T containing M and $\hat{c}$. We can define f on N in such a way that there be an f-preserving A-isomorphism between $\mathrm{acl}_T(A, c)$ and $\mathrm{acl}_T(A, \hat{c})$ which takes c to $\hat{c}$. We can embed N in a model $\hat{N}$ of T^*. Hence $\mathrm{tp}(c/A) = \mathrm{tp}(\hat{c}/A)$ by Lemma 1.4 and so c is not algebraic over A.
On the other hand we have $\mathrm{acl}_T(A) \subseteq \mathrm{acl}_{T^*}(A)$. $\square$*Lemma*

Definition 1.6 If T is simple we define for $B \subseteq C \subseteq \mathbf{C}^*$ and $\bar{a} \in \mathbf{C}^*$ the notion of T^**-independence* such that

$$\bar{a} \overset{*}{\underset{B}{\perp}} C \Leftrightarrow \bar{a} \underset{B}{\perp} C$$

where $\bar{a} \perp_B C$ is the notion of independence in the sense of T.

Lemma 1.7 *Let $M^* \models T^*$ and $\bar{a}, \bar{b}, c_1, c_2 \in \mathbf{C}^*$ such that*

$$\bar{a} \overset{*}{\underset{M^*}{\perp}} \bar{b}\ ,\ c_1 \overset{*}{\underset{M^*}{\perp}} \bar{a}\ ,\ c_2 \overset{*}{\underset{M^*}{\perp}} \bar{b},$$

$$\text{and } \mathrm{tp}(c_1/M^*) = \mathrm{tp}(c_2/M^*).$$

Then there is $c \in \mathbf{C}^*$ such that c realizes $\mathrm{tp}(c_1/\mathrm{acl}(M^*, \bar{a})) \cup \mathrm{tp}(c_2/\mathrm{acl}(M^*, \bar{b}))$ and

$$c \overset{*}{\underset{M^*}{\perp}} \bar{a}, \bar{b}\ .$$

Proof: First we restrict M^* to a model M of T. Then we have

$$\bar{a} \underset{M}{\perp} \bar{b}\ ,\ c_1 \underset{M}{\perp} \bar{a}\ ,\ c_2 \underset{M}{\perp} \bar{b}$$

and $\mathrm{tp}_T(c_1/M) = \mathrm{tp}_T(c_2/M)$. Since T is simple we can find $\hat{c} \in \mathbf{C}$ which realizes $\mathrm{tp}(c_1/\mathrm{acl}(M, \bar{a})) \cup \mathrm{tp}(c_2/\mathrm{acl}(M, \bar{b}))$ and

$$\hat{c} \underset{M}{\perp} \bar{a}, \bar{b}\ .$$

So in particular we have $\hat{c} \downarrow_{M,\bar{a}} \bar{b}$ and $\hat{c} \downarrow_{M,\bar{b}} \bar{a}$. By symmetry we get $\bar{a} \downarrow_{M,\bar{b}} \hat{c}$. Together with $\bar{a} \downarrow_{M} \bar{b}$ we get by transitivity $\bar{a} \downarrow_{M} \bar{b}, \hat{c}$ and in particular $\bar{a} \downarrow_{M,\hat{c}} \bar{b}$. From these independence properties we get (†)

$$\begin{aligned} \mathrm{acl}(M,\bar{a},\bar{b}) \cap \mathrm{acl}(M,\bar{a},\hat{c}) &= \mathrm{acl}(M,\bar{a}) \\ \mathrm{acl}(M,\bar{a},\bar{b}) \cap \mathrm{acl}(M,\bar{b},\hat{c}) &= \mathrm{acl}(M,\bar{b}) \\ \mathrm{acl}(M,\bar{a},\hat{c}) \cap \mathrm{acl}(M,\bar{b},\hat{c}) &= \mathrm{acl}(M,\hat{c}) \end{aligned}$$

Let N be a model of T which contains $M, \bar{a}, \bar{b}, \hat{c}$.
Since $\mathrm{tp}(\hat{c}/\mathrm{acl}(M,\bar{a})) = \mathrm{tp}(c_1/\mathrm{acl}(M,\bar{a}))$ we can find an automorphism $g_1 \in \mathrm{Aut}(\mathbf{C}/\mathrm{acl}(M,\bar{a}))$ such that $g_1(\hat{c}) = c_1$. In the same way we can find $g_2 \in \mathrm{Aut}(\mathbf{C}/\mathrm{acl}(M,\bar{b}))$ such that $g_2(\hat{c}) = c_2$.
Now we can expand N to an L^+ structure N^+ by defining f on N in the following way:

$$\begin{array}{rll} f^{N^+}(\bar{y}) = f^{\mathrm{acl}(M,\bar{a},\bar{b})}(\bar{y}) & \text{for} & \bar{y} \subseteq \mathrm{acl}(M,\bar{a},\bar{b}) \\ f^{N^+}(\bar{y}) = g_1^{-1}(f^{\mathbf{C}^+}(g_1(\bar{y}))) & \text{for} & \bar{y} \subseteq \mathrm{acl}(M,\bar{a},\hat{c}) \text{ such that there is} \\ & & y_1 \in \bar{y} \setminus \mathrm{acl}(M,\bar{a}) \\ f^{N^+}(\bar{y}) = g_2^{-1}(f^{\mathbf{C}^+}(g_2(\bar{y}))) & \text{for} & \bar{y} \subseteq \mathrm{acl}(M,\bar{b},\hat{c}) \text{ such that there are} \\ & & y_1 \in \bar{y} \setminus \mathrm{acl}(M,\bar{b}) \text{ and } y_2 \in \bar{y} \setminus \mathrm{acl}(M,\hat{c}) \\ f \text{ arbitrary} & & \text{everywhere else} \end{array}$$

By (†) this is well defined. Since T^* is the model completion of T^+ we can find $\hat{N}^* \supseteq N^+$ such that $\hat{N}^* \models T^*$. Since $N^* \prec \mathbf{C}^*$ we can find by homogenity of $\mathbf{C}^*$ the required c. $\Box$*Lemma*

Proposition 1.8 *If the theory T is (super-)simple then every completion of T^* is (super-) simple.*

Proof: This is because our notion of independence in T^* comes from the one of T and we also have the independence property over models by Lemma 1.7. Hence every completion of T^* is (super-)simple if T is (super-)simple using Theorem 4.2. from [3]. $\Box$*Proposition*

Proposition 1.9 *If T is stable and (weakly) eliminates imaginaries, so does every completion of T^**

Proof: The proof is identical to the proof of 2.9. in [2], if we replace $T_{P,S}$ by T^* and (M,P) by (M,f). $\Box$*Proposition*

What about adding more than one algebraic Skolem function?

Lemma 1.10 *If T is algebraically bounded, then T^* is also algebraically bounded.*

Proof: This is Corollary 3 from [4]. $\Box$

Remark 1.11 So we can add an algebraic Skolem function to a completion $\hat{T}^*$ of T^* and we will get a new expansion T_2^+ with a model-companion T_2^*. If T is simple then every completion of T_2^* is simple. We can repeat this n-many times. Note that we have to take a completion at the beginning of each step.

We can even add recursively, infinitely many Skolem functions:

Theorem 1.12 *Let T be simple. Let δ be an ordinal and T_δ^* be the model-completion of T after adding recursively δ many algebraic Skolem functions. Then every completion of T_δ^* is simple.*

Proof: By induction.
After the previous Remark we may assume that δ is an limit ordinal. Then $T_\delta^+ = \bigcup_{i<\delta} \hat{T}_i^*$ where $\hat{T}_i^*$ is a completion of T_i^*. So T_δ^+ is complete and model-complete. If T_δ^+ is not simple then there is $\phi(\bar{x}, \bar{y})$ which has the tree property. But then there is an $i < \delta$ and $\hat{T}_i^*$ such that $\phi(\bar{x}, \bar{y})$ is in this theory. But then $\hat{T}_i^*$ cannot be simple, which is a contradiction. $\Box Theorem$

So we can add for every algebraic formula its Skolem function and iterate and will get a simple theory in which algebraic closure is equal to definable closure. In particular:

Corollary 1.13 *The theory of algebraically closed fields (in any characteristic) has an expansion which has a model-completion such that every completion of this model-completion is supersimple of SU-rank 1 with algebraic closure equal to definable closure.*

2 Adding an algebraic Skolem function and categoricity

In this section we are going to examine whether categoricity is preserved under taking algebraic Skolem expansions.

Theorem 2.1 *If T is ω-categorical then every completion of T^* is ω-categorical.*

Proof: Let $\tilde{T}$ be a completion of T^*. Let A be a finite subset of $\tilde{\mathbf{C}}$. We may assume that A is algebraically closed. By Ryll-Nardzewski's theorem there are only finitely many 1-types over A in T since T is ω-categorical.
We have to show that there are also only finitely many 1-types over A in $\tilde{T}$. So let $b \in \tilde{\mathbf{C}}$. Then $\mathrm{acl}_T(A,b)$ is finite by ω-categoricity of T. Hence there are only finitely many different possibilities of defining the Skolem function f on $\mathrm{acl}_T(A,b)$. By Lemma 1.4 $\mathrm{tp}_T(b/A) = \mathrm{tp}_T(\hat{b}/A)$ if there is an f-preserving A-automorphism between $\mathrm{acl}_T(A,b)$ and $\mathrm{acl}_T(A,\hat{b})$ which takes b to $\hat{b}$. So there are only finitely many different expansions of $\mathrm{tp}_T(b/A)$ to a $\tilde{T}$-type. So there are only finitely many 1-types over A in $\tilde{T}$ and $\tilde{T}$ is ω-categorical. $\Box$*Theorem*

We cannot keep ω-categoricity in general if we add infinitely many algebraic Skolem functions (for example if we add an algebraic Skolem function for the same formula infinitely often). So in this case we have to be more carful about adding algebraic Skolem functions:

Theorem 2.2 *Let T be ω-categorical. Then there is a Skolem expansion which has a model-completion such that every completion is ω-categorical with algebraic closure equal to definable closure.*

Proof: We will inductively add for every $n < \omega$ finitely many n-ary algebraic Skolem function.
For every n there are by the Ryll-Nardzewski Theorem only finitely many inequivalent formulas $\psi_0(\bar{y},x), \psi_1(\bar{y},x), \ldots, \psi_m(\bar{y},x)$ in T where $\bar{y}$ consists of n distinct elements such that for every $\psi_i(\bar{y},x)$ there is $\bar{a} \in \mathbf{C}$ with $\{b \in \mathbf{C} \mid \mathbf{C} \models \psi_i(\bar{a},b)\}$ is finite. We assume further that every $\psi_i(\bar{y},x)$ contains the information that all elements of $\bar{y}$ are distinct.
Let $\psi(\bar{y},x)$ be such a formula and n_ψ its algebraic bound in x. Then we add inductively finitely many Skolem functions $f_1, f_2, \ldots, f_{n_\psi}$ where $f_j(\bar{y})$ is the algebraic Skolem function for the formula

$$\psi(\bar{y},x) \wedge \bigwedge_{i<j} \neg x = f_i(\bar{y}).$$

We do that for every formula $\psi_0(\bar{y},x), \psi_1(\bar{y},x), \ldots, \psi_m(\bar{y},x)$. So we add only finitely many n-ary Skolem functions. Let now T^+ be the Skolem expansion of all these new formulas and T^* its model-completion. Let $\tilde{T}$ a completion of T^*.
Then in $\tilde{T}$ algebraic closure is equal to definable closure. First note that for all $A \subseteq \tilde{\mathbf{C}}$ we have $\mathrm{acl}_{\tilde{T}}(A) = \mathrm{acl}_T(A)$ by Lemma 1.5. If now $b \in \mathrm{acl}_T(A)$ then there is $\psi(\bar{y},x)$ and $\bar{a} \in A$ such that

$$\mathbf{C} \models \psi(\bar{a},b) \wedge \exists^{<n_\psi} x\, \psi(\bar{a},x)$$

But then by construction there is a Skolem function $f(\bar{y})$ such that $f(\bar{a}) = b$ and $b \in \mathrm{dcl}(A)$.
Also, $\tilde{T}$ is ω-categorical. Let A be a finite subset of $\tilde{\mathbf{C}}$ and $\bar{b} \in \tilde{\mathbf{C}}$. Then $\mathrm{acl}_{\tilde{T}}(A, b)$ has n many elements. But there are only finitely many new functions of arity $\leq n$ and for all of then there are only finitely many ways to define them on $\mathrm{acl}_T(A, b)$. But this determines $\mathrm{tp}_{\tilde{T}}(A, b)$. So there are only finitely many expansions of $\mathrm{tp}_T(b/A)$ to a $\tilde{T}$-type. Since there are only finitely many 1-types over A in T there are also only finitely many 1-types over A in $\tilde{T}$. So $\tilde{T}$ is ω-categorical by the Ryll-Nardzewski theorem. $\Box$*Theorem*

Not suprisingly, we cannot keep $\aleph_1$-categoricity in general, since we cannot always avoid getting a two cardinal formula by adding an algebraic Skolem function.

Let now T be the theory of algebraically closed fields of a fixed characterisic not equal to 2. Then T is $\aleph_1$-categorical. Then even if we add the algebraic Skolem function $f(x)$ for the formula $x = y^2$ the model completion T^* will not be $\aleph_1$-categorical:

Lemma 2.3 *The theory of the random graph is interpretable in T^*.*

Proof: Let $M^* \models T^*$.
We may add a 2-ary relation symbol R to the language of T^* such that for all x, y we have
$$Rxy \;\Leftrightarrow\; f(x) \cdot f(y) = f(x \cdot y) \wedge x \neq y$$
since this relation is 0-definable.
We will show that R defines a model of the random graph on $M^* \setminus \{0,1\}$.
Since R is symmetric it defines an undirected graph on the elements of $M^* \setminus \{0,1\}$.
Let $\bar{v} = v_0, v_1, ..., v_{n-1}$, $\bar{w} = w_0, w_1, ..., w_{m-1} \in M^* \setminus \{0,1\}$ such that $\bar{v} \cap \bar{w} = \emptyset$.
Let $\phi(x, \bar{v}, \bar{w})$ the formula
$$\bigwedge_{i<n} Rxv_i \wedge \bigwedge_{j<m} \neg Rxw_j \wedge \neg x = 0 \wedge \neg x = 1$$
It is enough to show that $M^* \models \exists x \phi(x, \bar{v}, \bar{w})$.
Let $N \supseteq M$ a model of T and $a \in N \setminus M$. Now we define f on N in the following way

- $f^{N^+}(y) = f^{M^*}(\bar{y})$ for $y \in M^*$,
- $f^{N^+}(a)$ arbitrary,
- $f^{N^+}(v_i \cdot a) = f^{N^+}(v_i) \cdot f^{N^+}(a)$ for all $i < m$,

- $f^{N^+}(w_j \cdot a) = (-1) \cdot f^{N^+}(w_j) \cdot f^{N^+}(a)$ for all $j < n$,
- $f^{N^+}(y)$ arbitrary everywhere else

This is well defined since $v_0 \cdot a, v_1 \cdot a, ..., v_{m-1} \cdot a, w_0 \cdot a, w_1 \cdot a, ..., w_{n-1} \cdot a$ are all distinct and in $N \setminus M \cup \{a\}$.
So $N^+ \models T^+$ and $N^+ \models \phi(a, \bar{v}, \bar{w})$. Now let $\hat{N}^* \supseteq N^*$ such that $\hat{N}^* \models T^*$.
Then $\hat{N}^* \models \exists x \phi(x, \bar{v}, \bar{w})$.
Hence $M^* \models \exists x \phi(x, \bar{v}, \bar{w})$ since $M^* \preceq \hat{N}^*$. $\Box$*Lemma*

But then T^* is unstable and can not be $\aleph_1$-categorical.

3 Adding a nonalgebraic Skolem function

Let T be simple, model-complete and algebraically bounded. If we just add the Skolem function for a formula $\psi(\bar{y}, x)$ then its model-completion need not be simple:

Lemma 3.1 *Let $\psi(x, y, z)$ such that*

1. *There are a, b, c such that $\mathbf{C} \models \psi(a, b, c)$.*
2. *For all $n < \omega$, b_i, c_i for $i \leq n$, and a if for all $i \leq n$ we have $\mathbf{C} \models \psi(a, b_i, c_i)$, then the set $\{a' \mid \mathbf{C} \models \bigwedge_{i=0}^{n} \psi(a', b_i, c_i)\}$ is infinite.*
3. *For all a, b, c if $\mathbf{C} \models \psi(a, b, c)$ then the set $\{b' \mid \mathbf{C} \models \psi(a, b', c)\}$ is infinite.*
4. *For all a, b, c if $\mathbf{C} \models \psi(a, b, c)$ then the set $\{c' \mid \mathbf{C} \models \psi(a, b, c')\}$ is infinite.*

Add the Skolem function $f(x, y)$ for $\psi(x, y, z)$ to get T^+. Then the model-completion T^* of T^+ is not simple.

Remark 3.2 The formula $x = x \wedge y = y \wedge z = z$ satisfies for example the conditions of Lemma 3.1.

Proof of Lemma 3.1: We will show that the formula $f(x, y) = z$ has the tree property in T^*, i.e. we can find a set of elements $\{b_\nu, c_\nu \mid \nu \in \omega^\omega\}$ such that for each $\nu \in \omega^\omega$ the set $\{f(x, b_{\nu|_l}) = c_{\nu|_l} \mid l < \omega\}$ is consistent but the set $\{f(x, b_{\nu^\wedge i}) = c_{\nu^\wedge i} \mid i < \omega\}$ is 2-inconsistent.

If the tree is constructed up to level $n < \omega$ then we take for every ν such that $|\nu| = n$ a model $M^+ \models T^+$ such that $b_{\nu|_l}$, $c_{\nu|_l} \in M^+$ for all $l \leq |\nu|$. We denote with M the restriction of M^+ to the language of T. We can find a model $N \supseteq M$ of T such that there is an $a \in N \setminus M$ with

$$\models \bigwedge_{l \leq |\nu|} \psi(a, b_{\nu|_l}, c_{\nu|_l}).$$

This is possible as there are infinitely many solutions for the formula $\bigwedge_{l \leq |\nu|} \psi(x, b_{\nu|_l}, c_{\nu|_l})$ in M since $\{f(x, b_{\nu|_l}) = c_{\nu|_l} \mid l \leq |\nu|\}$ is consistent and by 2).
In particular we have $\models \psi(a, b_{\nu|_1}, c_{\nu|_1})$. By 3) there are infinitly many b such that $\models \psi(a, b, c_{\nu|_1})$. So we can find a $b \in N \setminus M$ such that

$$\models \psi(a, b, c_{\nu|_1}).$$

Using 4) we can find a $c \in N \setminus M$ such that

$$\models \psi(a, b, c)$$

(For the first step where $n = 0$ we can find by 1) a, b, c in a model N of T such that $\models \psi(a, b, c)$. We do not need M for this step.)

By 2) we can find distinct $a_i \in N \setminus M$ for $i < \omega$ such that

$$\models \psi(a_i, b, c) \wedge \bigwedge_{l \leq |\nu|} \psi(a_i, b_{\nu|_l}, c_{\nu|_l}).$$

Using 4) we can find distinct $c_i \in N \setminus M$ for $i < \omega$ such that for every $i < \omega$ we have

$$\models \psi(a_i, b, c_i).$$

Now we can define f on N in the following way to get $N^+ \models T^+$:

$$\begin{aligned} f^{N^+}(y_1, y_2) &= f^{M^+}(y_1, y_2) \text{ for } y_1, y_2 \in M^+ \\ f^{N^+}(a_i, b_{\nu|_l}) &= c_{\nu|_l} \text{ for } i < \omega\,,\ l \leq |\nu| \\ f^{N^+}(a_i, b) &= c_i \text{ for } i < \omega \end{aligned}$$

and f arbitrary everywhere else.
If we set now $b_{\nu^\wedge i} = b$ and $c_{\nu^\wedge i} = c_i$ for all $i < \omega$ then for all $i < \omega$ the set $\{f(x, b_{\nu|_i}) = c_{\nu|_l} \mid l \leq |\nu|\} \cup \{f(x, b_{\nu^\wedge i}) = c_{\nu^\wedge i}\}$ is consistent as it is satisfied by a_i but the set $\{f(x, b_{\nu^\wedge i}) = c_{\nu^\wedge i} \mid i < \omega\}$ is 2-inconsistent.

So $f(x, y) = z$ has the tree property in T^+ and also in T^* since this is the model completion of T^+ and the formulas are quantifier free. $\Box$*Lemma*

4 Adding a unary generic function

Let T be a complete theory in any language such that T has quantifier elimination and the algebraic closure in T is trivial, that is for all $A \subseteq \mathbf{C}$ we have $\mathrm{acl}(A) = A$.

Definition 4.1 We write $\exists^{\geq \omega} x \phi(x)$ for the set $\{\exists^{\geq n} x \phi(x) \mid n < \omega\}$.

Now we add a new unary function symbol f. Let $\varpi \subseteq \{1, 2, 3, 4, 5...\}$. The formulas in T_f^{ϖ} consist of:

1. T
2. For any formula $\psi(x, \bar{w})$ we have infinitely many formulas expressing:
$$\forall \bar{w} (\exists x (x \notin \bar{w} \wedge \psi(x, \bar{w})) \to \forall y \exists^{\geq \omega} x (f(x) = y \wedge \psi(x, \bar{w})))$$
3. Let $n \in \varpi$ and $\psi(x_0, x_1, ..., x_{n-1}, \bar{w})$ be any formula:
$$\begin{aligned} \forall \bar{w} \qquad & \exists x_0\, x_1 \ldots x_{n-1} \Big(\bigwedge_{0 \leq i < j < n} x_i \neq x_j \wedge \bigwedge_{i=0}^{n-1} x_i \notin \bar{w} \wedge \psi(x_0, x_1, ..., x_{n-1}, \bar{w}) \Big) \to \\ & \exists^{\geq \omega} x \Big(\bigwedge_{0 \leq i < j < n} f^i(x) \neq f^j(x) \wedge f^n(x) = x \wedge \psi(x, f(x), ..., f^{n-1}(x), \bar{w}) \Big) \Big) \end{aligned}$$
4. For all $n \notin \varpi$ we add
$$\forall x\, f^n(x) \neq x$$

Remark 4.2 Note that T_f^{ϖ} is the model completion of axioms 1 and 4, which will follow from Lemma 4.4.

Lemma 4.3 *T_f^{ϖ} is consistent.*

Proof: A model of T_f^{ϖ} can be constructed as the union of an infinite chain of structures. $\square$*Lemma*

Lemma 4.4 *T_f^{ϖ} has elimination of quantifiers.*

Proof: Let M_1 and M_2 be models of T_f^{ϖ} and A a common substructure. Let $\phi(x, \bar{y})$ be a conjunction of basic formulas and $\bar{a} \in A$. Assume that there is a $b \in M_1$ such that $M_1 \models \phi(b, \bar{a})$. We have to show that there is a $c \in M_2$ such that $M_2 \models \phi(c, \bar{a})$.
We may assume that for every $n < \omega$ and for every $a \in \bar{a}$ such that $f^n(a)$ appears in $\phi(x, \bar{a})$ we have $f^n(a) \in \bar{a}$ since $\phi(x, \bar{a})$ has finite length and $f^n(a) \in A$.

Case 1: There is $n < \omega$ and $a \in \bar{a}$ such that $f^n(a) = b$.
Then $b \in A \subseteq M_2$ and $M_2 \models \phi(b, \bar{a})$.

Case 2: There is $n < \omega$ and $a \in \bar{a}$ such that $f^n(b) = a$.
Let this n be minimal. Let $\psi(z_0, z_1, \ldots, z_{n-1}, \bar{a})$ be the formula $\phi(x, \bar{a})$ where each $f^i(x)$ is replaced by z_i for $i < n$. We may assume that $\psi(\bar{z}, \bar{a})$ contains the information that all elements of $(\bar{z}, \bar{a})$ are distinct.
Then $\psi(\bar{z}, \bar{a})$ is equivalent to a formula in the language of T and $M_1 \models \exists \bar{z} \psi(\bar{z}, \bar{a})$.
Hence $M_2 \models \exists \bar{z} \psi(\bar{z}, \bar{a})$.
Now we can inductively find elements $c_{n-1}, c_{n-2}, \ldots, c_1, c_0$ in M_2 such that

- $f(c_{n-1}) = a$,
- $f(c_i) = c_{i+1}$ for all $i < n-1$,
- $M_2 \models \exists \bar{z}_{i-1}, \ldots, z_0 \psi(z_0, \ldots, z_{i-1}, c_i, c_{i+1}, \ldots, c_{n-1}, \bar{a})$.

If we have already found $c_{n-1}, \ldots, c_i$ then

$$M_2 \models \exists \bar{z}_{i-1}, \ldots, z_0 (z_{i-1} \notin (\hat{c}_i, \ldots \hat{c}_{n-1}, \bar{a}) \wedge \psi(z_0, \ldots, z_{i-1}, c_i, c_{i+1}, \ldots, c_{n-1}, \bar{a}),$$

and we can find the required $\hat{c}_{i-1}$ by 2) of the axioms of T_f^{ϖ}.

Then $M_2 \models \phi(c_0, \bar{a})$.

Case 3: There is no $n < \omega$ and $a \in \bar{a}$ such that $f^n(a) = b$ or $f^n(b) = a$.
Since $\phi(x, \bar{y})$ has finite length there is an $n < \omega$ such that $f^n(x)$ appears in $\phi(x, \bar{y})$ but $f^m(x)$ does not appear in $\phi(x, \bar{y})$ for all $m > n$.

Case 3.1: There are $0 \leq i < j \leq n$ such that $f^i(b) = f^j(b)$.
Let i and j be minimal. We may assume that $j = n$.
Similarly to Case 2 let $\psi(z_0, \ldots, z_i, \ldots, z_{n-1}, \bar{a})$ be the formula $\phi(x, \bar{a})$ where each $f^k(x)$ is replaced by z_k for $k < n$ and $\psi(\bar{z}, \bar{a})$ contains the information that all elements of $(\bar{z}, \bar{a})$ are distinct.
Then $M_2 \models \exists z_{n-1}, \ldots, z_i \, \exists z_{i-1}, \ldots, z_0 \psi(z_{n-1}, \ldots, z_i, z_{i-1}, \ldots, z_0, \bar{a})$. So by the axioms of 3) we can find $c_{n-1}, \ldots, c_i \in M_2$ such that

- $f(c_k) = c_{k+1}$ for all $i \leq k < n-1$,
- $f(c_{n-1}) = c_i$,
- $M_2 \models \exists \bar{z}_{i-1}, \ldots, z_0 \psi(z_0, \ldots, z_{i-1}, c_i, \ldots, c_{n-1}, \bar{a})$.

Then we can find in the same way like in Case 2 elements $c_{i-1}, \ldots, c_0$ in M_2 such that

- $f(c_{i-1}) = c_i$,
- $f(c_k) = c_{k+1}$ for all $k < i-1$,
- $M_2 \models \psi(c_0, \ldots, z_{i-1}, c_i, \ldots, c_{n-1}, \bar{a})$.

Hence $M_2 \models \phi(c_0, \bar{a})$.

Case 3.2: There are no $0 \leq i < j \leq n$ such that $f^i(b) = f^j(b)$.
Here we can find again $c_0 \in M_2$ such that $M_2 \models \phi(c_0, \bar{a})$ like in Case 2.

As these are all cases T_f^{ϖ} has elimination of quantifiers. $\Box$*Lemma*

Lemma 4.5 *If T is stable then also T_f^{ϖ} is stable.*

Proof: Let $\lambda \geq \aleph_0$ such that $\lambda = \lambda^{\omega}$ and T is λ-stable. Let A be a subset of $\mathbf{C}$ of size λ.
It is enough to show that there are no more than λ many 1-types over A in T_f^{ϖ}. We only have to look at quantifier free types since T_f^{ϖ} has elimination of quantifiers by Lemma 4.4.
Let $b \in \mathbf{C}$. We will count the possibilities for $\mathrm{tp}_{T_f^{\varpi}}(b/A)$. We may assume that A is algebraically closed since $|\mathrm{acl}(A)| = |\{f^n(a) \mid a \in A\,,\, n < \omega\}| = |A| = \lambda$.

Case 1: $b \in A$.
There are only $\lambda = |A|$ many such types.

Case 2: There is $1 \leq n < \omega$ and $a \in A$ such that $f^n(b) = a$.
Let this n be minimal. If for $c \in \mathbf{C}$ also $f^n(c) = a$ then $\mathrm{tp}_{T_f^{\varpi}}(c/A) = \mathrm{tp}_{T_f^{\varpi}}(b/A)$ if $\mathrm{tp}_T(c, f(c), ..., f^{n-1}(c)/A) = \mathrm{tp}_T(b, f(b), ..., f^{n-1}(b)/A)$. By λ-stability of T there are not more than λ many T-n-types over A.
So there are no more than $\lambda \cdot \lambda = \lambda$ many such T_f^{ϖ}-1-types over A.

Case 3: There are $0 \leq i < j < \omega$ such that $f^i(b) = f^j(b)$.
Let i and j be minimal. Then again $\mathrm{tp}_{T_f^{\varpi}}(b/A)$ is determined by $\mathrm{tp}_T(b, f(b), ...,$

$f^{j-1}(c)/A)$ and there are at most λ many different non algebraic T-j-types over A.
Since we have ω many choices for i and j there are no more than $\lambda \cdot \omega = \lambda$ many such T_f^{ϖ}-1-types over A.

Case 4: There are no $0 \leq i < j < \omega$ or $a \in A$ such that $f^i(b) = f^j(b)$ or $f^i(b) = a$.
Then $\mathrm{tp}_{T_f^{\varpi}}(b/A)$ is determined if we know for every $n < \omega$ the type $\mathrm{tp}_T(f^n(b)/f^{n-1}(b), ..., f(b), b, A)$. But since $|f^{n-1}(b), ..., f(b), b, A| = \lambda$ there are at most λ many different possibilities for every $f^n(b)$.
Hence there are at most $\lambda^{\omega} = \lambda$ many such types.

So in all four cases we do not get more than λ many different T_f^{ϖ}-1-types over A and hence there are no moren than λ many different T_f^{ϖ}-1-types over A in total. Hence T_f^{ϖ} is λ-stable and in particular stable. $\square$*Lemma*

Remark 4.6 *If T is ω-stable or small then T_f^{ϖ} is not necessarily ω-stable or small.*

For example let T be the theory of pure sets with a unary predicate P which is infinite and coinfinite. Then T is ω-stable and small. But T_f^{ω} is not, because for every $N \subseteq \omega$ we can define the 1-type

$$\{f^i(x) \neq f^j(x) \mid i < j < \omega\} \cup \{P(f^n(x)) \mid n \in N\} \cup \{\neg P(f^n(x)) \mid n \notin N\}.$$

Hence we have $2^{\aleph_0}$ many different 1-types over the empty set.

Lemma 4.7 *If T is simple then also T_f^{ϖ} is simple.*

The proof goes by counting types using some results from [1].

Definition 4.8 For every pair κ, λ of (infinite) cardinals let $\mathrm{N}T(\kappa, \lambda)$ be the supremum of the cardinalities $|P|$ of families P which consist of pairwise incompatible partial types of size $\leq \kappa$ over a set of cardinality $\leq \lambda$.

Fact 4.9 *T is simple if and only if for all κ, λ we have $\mathrm{NT}(\kappa, \lambda) \leq \lambda^{|T|} + 2^{\kappa}$.*

This is Theorem 2.8. from [1].

Proof of Lemma 4.7: It is enough to consider 1-types. Let A be a set of size $\leq \lambda$ in $\mathbf{C} \models T_f^{\varpi}$. By Fact 4.9 there a $\mu \leq \lambda^{|T|} + 2^{\kappa}$ many pairwise incompatible partial T-types of size $\leq \kappa$ over A. Using the same arguments and cases as in Lemma 4.5 there are $\leq \mu + \mu \cdot \mu + \mu \cdot \omega + \mu^\omega \leq \mu^\omega \leq (\lambda^{|T|} + 2^\kappa)^\omega = \lambda^{|T|} + 2^\kappa$ many pairwise incompatible partial T_f^ϖ-types of size $\leq \kappa$ over A. Hence we have $\mathrm{N}T(\kappa, \lambda) \leq \lambda^{|T|} + 2^\kappa$ for T_f^ϖ and T_f^ϖ is simple by Fact 4.9. $\square$*Lemma*

References

[1] Enrique Casanovas, *The number of types in simple theories*, Annals of Pure and Applied Logic **98** (1999), 69–86.

[2] Z. Chatzidakis, A. Pillay, *Generic structures and simple theories*, Annals of Pure and Applied Logic **95** (1998), 71–92.

[3] Byunghan Kim, Anand Pillay, *Simple theories*, Annals of Pure and Applied Logic **88** (1997), 149–164.

[4] Peter M. Winkler, *Model-Completeness and Skolem Expansions*, Model Theory and Algebra - A Memorial Tribute to Abraham Robinson, (D.H. Saracino and V.B. Weispfenning, eds.), Lecture Notes in Mathematics, no. 498, Springer-Verlag, 1975, pp. 408–463.

[5] B.I. Zilber, *The structure of models of uncountably categorical theories* , Proceedings of the International Congress of Mathematicians, Vol.1,2 (Warsaw, 1983), Warsaw (1984), 359–368.

Author's address:
School of Mathematics, UEA, Norwich NR4 7TJ, England
e-mail: `H.Nubling@uea.ac.uk`