Continuous Cohomology of Permutation Groups on Profinite Modules

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Abstract. We investigate the continuous cohomology of infinite permutation groups on modules whose topology is profinite. To obtain acyclics we expand the class of modules to include those which are directed unions of their profinite submodules. As an application we give a criterion which implies finiteness of the continuous cohomology groups on finitely generated profinite modules for some familiar permutation groups.

Introduction

By a permutation group Γ on a set Ω we mean a subgroup of $\mathrm{Sym}(\Omega)$, the symmetric group on Ω (we may occasionally refer to 'the permutation group $(\Omega;\Gamma)$'). We regard Γ as a topological group by taking pointwise stabilizers of finite sets as a base of open neighbourhoods of the identity: this is simply the topology of pointwise convergence. As is well-known, the closed subgroups of $\mathrm{Sym}(\Omega)$ are precisely the automorphism groups of first-order structures with domain Ω.

A permutation group is compact if and only if it is closed and all of its orbits are finite. In this case it is a closed subgroup of a direct product of finite groups, and is therefore profinite. Thus it is quite natural to consider profinite modules for permutation groups and the corresponding extension problem (for topological groups). In fact, the extension problem can also be seen at the level of permutation groups. It is easy to see that if K is a compact normal subgroup of a permutation group $\Gamma \leq \mathrm{Sym}(\Omega)$, then the K-orbits form the classes of a Γ-invariant equivalence relation $\sim$ on Ω. Moreover, if Γ is closed, then the permutation group G induced on $\Delta = \Omega/\sim$ by Γ is closed. We say that the surjective map $(\Omega;\Gamma) \to (\Delta;G)$ in the category of permutation groups is a *finite cover* of $(\Delta; G)$. The *cover problem* is the extension problem: describe the possibilities for $(\Omega; \Gamma)$, given $(\Delta; G)$.

Questions of this nature arise naturally in model theory: of course, one usually has extra hypotheses adding to the tractability of the problem. Not surprisingly, the case where K is abelian has a special role. Here one has a continuous action of G on K making it into a topological G-module. In this case, it has been useful to consider, for profinite G-modules N, the cohomology group $H^1_c(G, N)$: continuous derivations $G \to N$ modulo inner derivations. The reader should consult [**4, 6**] for details, applications and further references.

Our aim in this paper is to introduce higher cohomology groups $H^n_c(G, N)$ and to develop some basic cohomological machinery for them. Of course, as far as the extension problem is concerned, the main interest is in H^2_c. But there seems little simplicfication to be gained by ignoring the higher order groups.

A slightly unexpected feature of our approach is that in order to construct acyclic modules (that is, modules M where $H^n_c(G, M) = 0$ for $n \geq 1$), we extend our category to include *weakly profinite* modules: countable directed unions of profinite modules, topologised with the weak topology from these. We can then define modules coinduced from the identity subgroup of G and, in certain cases, prove injectivity properties for these.

If N is a profinite G-module, we define $H^*_c(G, N)$ to be the cohomology of the cochain complex arising from the usual differential restricted to *continuous* $G^n \to N$. In Section 1 we show that this obeys the long exact sequence, and Shapiro's Lemma holds for profinite modules coinduced from an open subgroup of G. In Section 2 we prove that extensions in the category of permutation groups (as topological groups) are parametrised by the second continuous cohomology group: suggesting that our definition is 'correct.'

In Section 3 we derive some basic results about groups which are countable direct limits of profinite groups, and in Section 4 we extend our cohomology to include modules which are countable direct limits of profinite submodules.

This technology is applied in Section 5 to compute continuous cohomology groups on trivial (finite) modules. Theorem 5.10 gives a straightforward criterion (on the permutation group) which guarantees that these trivial modules are acyclic. In Corollary 5.13 this is used to show that for certain familiar infinite permutation groups the continuous cohomology groups on finitely generated profinite modules are finite.

1. Continuous cohomology

Suppose G is a topological group. A topological abelian group N with a continuous G-action $G \times N \to N$ is called a *continuous* G-module. Note that we always work with left modules.

Suppose N is a continuous G-module. For $n \geq 0$ let $C^n(G,N)$ be the additive group of functions $G^n \to N$ and $\tilde{C}^n(G,N)$ the subgroup consisting of the continuous functions. Let $d_n : C^n(G,N) \to C^{n+1}(G,N)$ be the usual differential, viz:

$$(d_n f)(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) +$$
$$\sum_{i=1}^{n} (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n).$$

Then $d_{n+1} d_n = 0$ and $d_n(\tilde{C}^n(G,N)) \subseteq \tilde{C}^{n+1}(G,N)$. For convenience, we let $d_{-1} : \{0\} \to C^0(G,N)$ be the zero function. Let $Z^n(G,N)$ be the kernel of d_n (the n-cocycles), and $\tilde{Z}^n(G,N)$ the kernel of d_n restricted to $\tilde{C}^n(G,N)$ (the continuous cocycles). Let $B^n(G,N)$ be the image of d_{n-1} (the n-coboundaries) and $\tilde{B}^n(G,N) = d_{n-1}(\tilde{C}^{n-1}(G,N))$. Abusing terminology, we refer to the latter as the continuous coboundaries. We then define $H^n(G,N) = Z^n(G,N)/B^n(G,N)$ as usual, and $\tilde{H}^n(G,N) = \tilde{Z}^n(G,N)/\tilde{B}^n(G,N)$. Note that $H^0(G,N) = \tilde{H}^0(G,N) = N^G$, the fixed points of G on N.

Lemma 1.1 (The long exact sequence). *Suppose* $0 \to A \xrightarrow{\iota} B \xrightarrow{\bar{}} C \to 0$ *is a short exact sequence of continuous G-modules in which the maps are continuous, open G-homomorphisms. Suppose further that there is a continuous section* $\gamma : C \to B$ *of* $B \xrightarrow{\bar{}} C$. *Then we have an exact sequence:*

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to \tilde{H}^1(G,A) \to \dots$$
$$\dots \to \tilde{H}^i(G,C) \to \tilde{H}^{i+1}(G,A) \to \tilde{H}^{i+1}(G,B) \to \tilde{H}^{i+1}(G,C) \to \dots$$

Proof. We believe this to be well-known: cf. ([**9**], 2.1), for example. □

We remark that the case of particular interest to us is where A, B, C are profinite. In this case, all we require is that the maps be continuous homomorphisms: openness follows from the compactness, and existence of continuous sections is well-known (cf. [**10**], 2.1., Proposition 1).

Definition 1.2 (Coinduction from an open subgroup). Suppose G is a topological group, H an open subgroup of G and M a continuous

H-module. The *coinduced module* $N = M{\Uparrow}_H^G$ is the additive group of H-equivariant functions $f : G \to M$ (that is, $f(hx) = hf(x)$ for all $h \in H$ and $x \in G$) with the G-action $(gf)(x) = f(xg)$ for $g, x \in G$. Note that there is a natural isomorphism (of abelian groups) $N \to M^{G/H}$ obtained by evaluating functions at a fixed set of right coset representatives. We topologise N using the product topology on $M^{G/H}$ (which is the same thing as considering $N \subseteq M^G$ with the product topology).

Lemma 1.3. *(i) With the above notation, $N = M{\Uparrow}_H^G$ is a continuous G-module.*
(ii) If $f : G^n \to N$ is continuous, then the function $\hat{f} : G^{n+1} \to M$ given by $\hat{f}(x_0, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)(x_0)$ is continuous.

Proof. (i) Suppose $x \in G$, O an open subset of M, $g \in G$ and $f \in N$ such that $(gf)(x) \in O$. It suffices to find open neigbourhoods T of g and Y of f such that if $g' \in T$ and $f' \in Y$ then $(g'f')(x) \in O$. Let $z = (gf)(x)$. As the action $H \times M \to M$ is continuous and H is open in G there exist an open neighbourhood $Z \subseteq M$ of z and an open neighbourhood $X \subseteq G$ of 1 with $X \subseteq H$ and $XZ \subseteq O$. Let $Y = \{f' : f'(xg) \in Z\}$ and $T = x^{-1}Xxg$. This works, as the reader can easily check.

(ii) Suppose $f(x_1, \ldots, x_n)(x_0) = y$ and Y is an open neighbourhood of y in M. As the action $H \times M \to M$ is continuous, there is H_0, an open neighbourhood of the identity in H, and Z, an open neighbourhood of y in M, with $H_0 Z \subseteq Y$. The composition of f with the evaluation map at x_0 is continuous, so there exist open neighbourhoods $X_1, \ldots, X_n$ of $x_1, \ldots, x_n$ in G such that $f(X_1, \ldots, X_n)(x_0) \subseteq Z$. Then $X_0 = H_0 x_0$ is an open neighbourhood of x_0 in G and

$$\hat{f}(X_0, X_1, \ldots, X_n) = f(X_1, \ldots, X_n)(X_0) = H_0 f(X_1, \ldots, X_n)(x_0) \subseteq Y.$$

□

Again, we believe that the following is well-known, but are unable to locate a proof of the result in this form. Versions of the result exist for other families of continuous modules (in particular where G is locally compact): see [**7**], for example. In the more familiar contexts, the proof goes by induction on n, using exactness of coinduction from H and acyclicity of modules coinduced from the trivial subgroup. We do not (yet) have the latter in our context.

Lemma 1.4 (Lemma of Eckmann-Faddeev-Shapiro). *Suppose G is a topological group and H an open subgroup of G. Let M be a continuous*

H-module and $N = M{\Uparrow}_H^G$. Then for all $n \in \mathbb{N}$:

$$\tilde{H}^n(G,N) \cong \tilde{H}^n(H,M).$$

Proof. The proof is an explicit calculation extracted from standard proofs in the literature. We omit the (purely formal) checking of various details.

By definition, $\tilde{H}^*(G,N)$ and $\tilde{H}^*(H,M)$ are the (co)homology of the cochain complexes $(\tilde{C}^n(G,N); d_n)$ and $(\tilde{C}^n(H,M); d'_n)$ (where primes here are just a notational convenience). We show that these complexes are homotopic.

Define $\rho_n : \tilde{C}^n(G,N) \to \tilde{C}^n(H,M)$ by restriction and evaluation at 1. Thus, if $f \in \tilde{C}^n(G,N)$ and $x_1,\ldots,x_n \in H$:

$$(\rho_n f)(x_1,\ldots,x_n) = f(x_1,\ldots,x_n)(1).$$

It is easy to check that

$$\rho_{n+1} d_n = d'_n \rho_n.$$

Next, we want to define maps $\phi_n : \tilde{C}^(H,M) \to \tilde{C}^n(G,N)$. Let $\tau : G \to G$ select a representative from each right coset of H in G. So $\tau(hx) = \tau(x)$ for $h \in H$ and $x \in G$. Also take $\tau(1) = 1$. As H is open in G, τ is continuous. Define $\theta : G \to H$ by $\theta(x) = x\tau(x)^{-1}$, and note that this is continuous and $\theta(hx) = h\theta(x)$ (for $h \in H$).

If $k \in \tilde{C}^n(H,M)$ define $f : G^n \to N$ by:

$$f(x_1,\ldots,x_n)(x) =$$
$$\theta(x)k(\theta(x)^{-1}\theta(xx_1), \theta(xx_1)^{-1}\theta(xx_1x_2), \ldots, \theta(xx_1\ldots x_{n-1})^{-1}\theta(xx_1\ldots x_n)).$$

One sees easily that $f(x_1,\ldots,x_n)$ is H-equivariant and f is continuous (using continuity of k, θ and the H-action on M). We define

$$\phi_n(k) = f.$$

A routine calculation shows that:

$$d_n\phi_n = \phi_{n+1}d'_n.$$

Moreover, it is easy to see that:

$$\rho_n\phi_n = \text{ identity on } \tilde{C}^n(H,M).$$

Thus to finish the proof we have to show that $\phi_n\rho_n$ is homotopic to ι_n, the identity on $\tilde{C}^n(G,N)$. We define maps $h_n : \tilde{C}^{n+1}(G,N) \to \tilde{C}^n(G,N)$ as follows. If $g \in \tilde{C}^{n+1}(G,N)$ and $x_1,\ldots,x_n,x \in G$ let:

$$(h_n g)(x_1,\ldots,x_n)(x) = g(x^{-1}\theta(x), \theta(x)^{-1}\theta(xx_1), \theta(xx_1)^{-1}\theta(xx_1x_2), \ldots)+$$
$$\sum_{j=1}^{n}(-1)^j g(x_1,\ldots,x_j,(xx_1\ldots x_j)^{-1}\theta(xx_1\ldots x_j), \theta(xx_1\ldots x_j)^{-1}\theta(xx_1\ldots x_{j+1}),\ldots).$$

Continuity of $h_n g$ follows from continuity of g, θ and Lemma 1.3(ii). A purely formal (if somewhat non-trivial) calculation then shows that:

$$d_{n-1}h_{n-1} + h_n d_n = \phi_n \rho_n - \iota_n,$$

which is as required. $\square$

If M, N are continuous G-modules, denote by $\mathrm{Hom}_G(M, N)$ the continuous G-homomorphisms from M to N. As usual we have:

Lemma 1.5 (Frobenius reciprocity). *If N is a continuous G-module, H an open subgroup of G and M a continuous H-module then*

$$\mathrm{Hom}_G(N, M{\Uparrow}_H^G) \cong \mathrm{Hom}_H(N, M).$$

Proof. If $f \in \mathrm{Hom}_G(N, M{\Uparrow}_H^G)$ define $\theta(f) : N \to M$ by $\theta(f)(n) = f(n)(1)$ (for $n \in N$). If $e \in \mathrm{Hom}_H(N, M)$ define $\gamma(e) : N \to M{\Uparrow}_H^G$ by $(\gamma(e)(n))(g) = e(gn)$ (for $n \in N$ and $g \in G$). We leave the reader to check that θ, γ give mutually inverse maps between $\mathrm{Hom}_G(N, M{\Uparrow}_H^G)$ and $\mathrm{Hom}_H(N, M)$. $\square$

2. Permutation groups and extensions

We denote by $\mathcal{PG}$ the class of Hausdorff topological groups whose topology is generated by a family of open subgroups (which we always take to be closed under finite intersections). Permutation groups with the topology defined in the introduction are examples here. Conversely if $G \in \mathcal{PG}$ then G is isomorphic to a subgroup of a symmetric group. Indeed, if $\{U_i\}$ is a family of open subgroups which generate the topology on G, then the permutation representation on the union of the left coset spaces $G \to \mathrm{Sym}(\coprod G/U_i)$ gives an isomorphism between G and a subgroup of this symmetric group. This is a closed subgroup precisely when G is complete with respect to the two-sided uniformity given by the U_i. Abusing terminology we refer to $\mathcal{PG}$ as the class of permutation groups.

Lemma 2.1. *Suppose G is a permutation group and M is a continuous G-module. If N is an open subgroup of M, then there is an open subgroup of N whose stabilizer in G is open.*

Proof. The preimage in $G \times M$ of N contains $(1, 0)$ and hence contains a product $H \times U$, where H is an open subgroup and U is an open neighborhood of 0. We may assume that $U \subseteq N$. The open subset HU generates an open subgroup P that is contained in N and is H-invariant. $\square$

Corollary 2.2. *Suppose G is a permutation group and M is a continuous profinite module.*

(1) *If N is an open subgroup of M then its stabilizer in G is open.*
(2) *The G-action on the dual group $M^\vee$ is continuous.*

PROOF. Let P be an open subgroup of N whose stabilizer H is open. Since P has finite index in M the centralizer in H of M/P is a closed subgroup of finite index in H. This centralizer is therefore open in G. Since the stabilizer of N contains the centralizer of M/P, the stabilizer of N is open also.

Now consider the action on $M^\vee$. Since M is compact $M^\vee$ is discrete. Let ϕ be any character. If H is the centralizer of $M/\ker(\phi)$ then the preimage in $G \times M^\vee$ of the open set $\{\phi\}$ contains the open set $H \times \{\phi\}$. Hence the G-action is continuous at ϕ. □

We refer to the properties in the conclusions above as *strong continuity* of G on M.

LEMMA 2.3. *Suppose Γ is a permutation group and N is a compact normal subgroup of Γ. Then there is a continuous, closed section $\phi : \Gamma/N \to \Gamma$ of the natural homomorphism $\Gamma \to \Gamma/N$. In particular, Γ is homeomorphic to $(\Gamma/N) \times N$.*

PROOF. By 'closed' here, we mean that ϕ maps closed sets to closed subsets of Γ. As ϕ is injective, this is the same as being a proper map. Let $G = \Gamma/N$. The proof follows that of Proposition 1.2.1 in [**10**].

We let $\mathcal{S}$ be the set of pairs (S, s) where S is a closed subgroup of N and $s : G \to (\Gamma : S)$ is a continuous closed section of the natural map $(\Gamma : S) \to G$ (where $(\Gamma : S)$ denotes the left coset space). This is ordered in the following way: $(S, s) \leq (S', s')$ iff $S' \leq S$ and s is the composition of s' with the natural map $(\Gamma : S') \to (\Gamma : S)$. Note that $\mathcal{S} \neq \emptyset$ (by taking $S = N$).

Suppose $\mathcal{S}'$ is a chain in $\mathcal{S}$: for convenience we write this as $\{(S_i, s_i) : i \in I\}$ (where I is an appropriate indexing set). Let $S = \bigcap_i S_i$. By ([**1**], III 7.2, Corollary 3), $(\Gamma : S)$ is naturally homeomorphic with $\varprojlim(\Gamma : S_i)$ (which is a closed subspace of the product $\prod(\Gamma : S_i)$). Moreover $s = \varprojlim s_i : G \to \varprojlim(\Gamma : S_i)$ is proper (by ([**1**], I 10.2, Corollary 4). Thus (S, s) is an upper bound for $\mathcal{S}'$ in $\mathcal{S}$.

By Zorn's Lemma and the above, there exists a maximal element (S, s) of $\mathcal{S}$. We claim $S = 1$ (and so we take $\phi = s$). Suppose not. Then there is an open subgroup Σ of Γ with $S_1 = \Sigma \cap S \neq S$. We may also assume that Σ is normalised by S. (This follows from the fact that we are dealing with a permutation group on some set Ω. The N-orbits on Ω are finite (by compactness of N) and pointwise stabilisers of finite unions of these form a base of open neighbourhoods of the identity in Γ and are all normalised by N.) For a contradiction, it is enough to prove that there is continuous closed map $t : (\Gamma : S) \to (\Gamma : S_1)$ with $t(gS)S = gS$ for all $g \in \Gamma$.

Note that ΣS is an open subgroup of Γ. Let $(g_i : i \in I)$ be a (normalised) system of left coset representatives for ΣS in Γ. Define $t' : (\Sigma : S) \to (\Gamma : S_1)$ by $t'(gS) = gS_1$. This is well-defined, injective and continuous. In fact it is also proper, as the natural map $\Gamma \to (\Gamma : S_1)$ is proper, by ([**1**], III 4.1, Corollary 2). Define $t_i : (g_i\Sigma : S) \to (\Gamma : S_1)$ for $i \in I$ by translating this: $t_i(g_i xS) = g_i xS_1$. The sets $(g_i\Sigma : S)$ are pairwise disjoint and form a clopen covering of $(\Gamma : S)$. The union t of the t_i is a continuous proper map $(\Gamma : S) \to (\Gamma : S_1)$ which is a section of $(\Gamma : S_1) \to (\Gamma : S)$. (To see that t is proper, note that it is injective so it is enough to show that it is a closed map. Its image is $\bigcup g_i(\Sigma : S_1)$. This is closed in $(\Gamma : S_1)$ as $\bigcup g_i\Sigma$ is a clopen subset of Γ. So it suffices to observe that t maps open subsets to open subsets of the image. This is true for t', and this is enough.) Then $(S_1, t \circ s) \in \mathcal{S}$ and is greater than (S, s): a contradiction.

For the final part, define a map $\theta : G \times N \to \Gamma$ by $\theta(g, n) = \phi(g)n$. This is a continuous bijection. It is the composition of maps

$$G \times N \xrightarrow{\phi \times \iota} \Gamma \times N \xrightarrow{\cdot} \Gamma.$$

The first of these is a closed map as ϕ is proper. Moreover the second is also a proper map as N is compact (use ([**1**] III 4.1, Proposition 1). So θ is a closed map, and therefore a homeomorphism. □

THEOREM 2.4. *Suppose G is a permutation group and N is a profinite G-module. Then there is a bijective correspondence between $\tilde{H}^2(G, N)$ and equivalence classes of extensions $1 \to N \to \Gamma \to G \to 1$, where the groups Γ are permutation groups, maps are continuous open, and equivalence is up to topological isomorphism.*

PROOF. Given a continuous 2-cocycle $h : G^2 \to N$ we define a group structure on the (topological) product $G \times N$ in the usual way. This gives a topological group Γ, and Corollary 2.2 means the topology on Γ is generated by open subgroups. Conversely, the above lemma shows such an extension Γ is, topologically, a product, and we obtain a continuous cocycle in the usual way. Finally, one verifies that equivalence of extensions corresponds to varying the cocycle by a continuous coboundary, and again, there is nothing new to be done here. □

We remark that Γ is complete iff it is representable as a closed permutation group iff G is complete iff G is representable as a closed permutation group.

3. Weakly profinite groups

In order to develop the cohomology theory further (and indeed to calculate some cohomology groups) we need a supply of acyclic modules. The class of continuous G-modules which are of most interest to us are profinite modules. These form an abelian category (with continuous homomorphisms), but there is no reason to suppose this category has enough injectives, or contains acyclic modules. To obtain the latter we enlarge the category by taking countable direct limits of profinite G-modules. In this section we develop some basic properties of such topologised groups. The extension of the cohomology theory to accommodate these *weakly profinite* G-modules, and constructions of acyclics are in subsequent sections.

We work with abelian groups M equipped with a topology (*a priori* this need not be a group topology). We say that M is *weakly profinite* if it is the union of a countable upwardly directed family $\{M_i\}_{i<\omega}$ of profinite subgroups with respect to which it carries the weak topology. This means that a subset of M is open precisely when its intersection with each M_i is open. Equivalently, a map from M into any space is continuous precisely when its restriction to each M_i is continuous. We refer to the M_i here as a *profinite system* for M.

So here each M_i is a topological group and of course the group operation and the topology on M are determined by their restrictions to the M_i. In particular, translation by any element of M is a homeomorphism of M. We show that a weakly profinite abelian group M is a topological group and its topology is independent of the profinite family used to define it.

LEMMA 3.1. *Suppose M is weakly profinite with respect to $\{M_i\}_{i<\omega}$. If K is a compact subset of M, then K is contained in one of the M_i.*

PROOF. Suppose not. Then $K \cap (M_{i+1} \setminus M_i)$ is non-empty for infinitely many $i < \omega$. For each such i, choose an element of this set, and let X be the set of these elements. So X is an infinite subset of K whose intersection with each M_i is finite. From the latter it follows that X is a closed, discrete subset of M, and therefore of K. But as K is compact, this is impossible. □

We say that a subset of a weakly profinite group $M = \bigcup_{i<\omega} M_i$ is *bounded* if it is contained in some M_i. The lemma says that compacta are bounded. Hence an equivalent definition is that the bounded subsets are those whose closure is compact. Similarly a map to M is bounded when its image is contained in a compact subset.

Lemma 3.2. *Closed subgroups and quotients of a weakly profinite group by closed subgroups are weakly profinite.*

Proof. Suppose M is weakly profinite with respect to a family $\{M_i\}_{i<\omega}$ and suppose N is closed in M. First we show that the relative topology on N is the weak topology with respect to the subgroups $N_i = N \cap M_i$. Clearly if $X \subseteq M$ is closed then $X \cap N_i$ is closed in N_i for all i, so $X \cap N$ is closed in N with the weak topology determined by the N_i. On the other hand, if $Y \subseteq N$ and $Y \cap N_i$ is closed in N_i for all i, then as N_i is closed in M we have that $Y \cap M_i$ is closed in M_i for all i. So Y is closed in M.

Next, consider the quotient map $p\colon M \to M/N$. We claim that the quotient topology is the same as the weak topology generated by the $p(M_i)$. Suppose $\bar{X}$ is a subset of M/N with the property that each $\bar{X} \cap p(M_i)$ is open in $p(M_i)$. Now

$$p^{-1}(\bar{X}) \cap M_i = p^{-1}(\bar{X} \cap p(M_i)) \cap M_i.$$

Since p is continuous this intersection is open in M_i. As M has the weak topology we conclude that $p^{-1}(\bar{X})$ is open. Hence $\bar{X}$ is open, by definition of the quotient topology. □

Lemma 3.3. *Suppose M, L are weakly profinite abelian groups and $\theta : M \to L$ is a continuous epimorphism. Then θ is an open map. In particular, if θ is an isomorphism, it is a homeomorphism.*

Proof. The induced map $M/\ker\theta \to L$ is continuous, so by the above lemma it suffices to prove the final statement. So assume θ is a continuous isomorphism. As L has the weak topology with respect to a family of profinite subgroups it suffices to show that θ^{-1} is continuous when restricted to one of these. The image of this is certainly closed in M so (again by the above lemma) we reduce to the case where L is profinite.

If M is also compact, then the result is well-known. So suppose M is the union of a chain of proper compact subgroups $\{M_i\}_{i<\omega}$. Clearly these must be of infinite index in M. Let $L_i = \theta(M_i)$. So L_i is a compact (in particular, closed) proper subgroup of L. It is of infinite index in L, so is nowhere dense. So we have that L is compact and the union of a countable chain of nowhere dense subsets. This contradicts Baire's Theorem ([**3**], XI Theorem 10.1, for example). □

Remarks 3.4. Thus the topology on a weakly profinite group M is determined by any countable chain of compact subgroups with union M. Lemma 3.1 says that any two such chains are cofinal in each other. Note also that we can define a weakly profinite group by a countable directed system of profinite submodules: any cofinal chain in such a system gives the same weak topology.

Suppose $\{M_i\}$ are topological groups, and let M be their (algebraic) direct sum. For each finite set of indices I let $M_I = \oplus_I M_i$, with the product topology. If we endow M with the weak topology generated by the M_I then we obtain what is sometimes called the *finite topology*. In general the finite topology is not a group topology. Nevertheless the category of topological abelian groups does admit coproducts — in general the coproduct is some rather mysterious topology on the algebraic direct sum. If the finite topology is a group topology then it is the coproduct topology. For more on this see [**8**].

Lemma 3.5. *Suppose that $\{M_i\}_{i<\omega}$ is a family of topological groups each of whose topologies is generated by open subgroups. If $M = \oplus M_i$ with the finite topology then M is a topological group.*

Proof. We show that the group operation $M \times M \to M$ is continuous. As translation by any particular element of M is a homeomorphism, it suffices to show that any open neighbourhood $X \subseteq M$ of the identity contains an open subgroup. But $X \cap M_j$ is open in M_j and so contains an open subgroup N_j of M_j: thus X contains $\oplus_j N_j$. This is easily seen to be open in M.

A similar argument shows that inversion is continuous. □

Corollary 3.6. *Suppose L is a weakly profinite abelian group. Then L is a topological group.*

Proof. Let $\{L_i\}_{i<\omega}$ be a system of profinite subgroups for L. Let $M = \oplus L_j$ with the finite topology and $\theta : M \to L$ the homomorphism which is the identity on each direct summand. Clearly the restriction of θ to any finite direct sum is continuous, so as M has the weak topology with respect to these, θ is continuous. As M, L are weakly profinite and θ is surjective, Lemma 3.3 shows that θ is an open map. Thus L is isomorphic to the quotient $M/\ker\theta$, and is therefore a topological group (as a quotient group of a topological group is a topological group). □

Remarks 3.7. It would have been more satisfactory to define weakly profinite groups as arbitrary (rather than countable) directed unions of profinite abelian groups. Indeed some of the results (in particular, Corollary 3.6) also hold in this more general context. However we were not able to prove Lemmas 3.1 and 3.3 in the wider context.

4. Weakly profinite modules and bounded cohomology groups

We now consider countable directed unions of profinite G-modules. We do not require that the G-action be continuous globally, but only on compact submodules. A simple example should clarify the situation. Let $G = \mathrm{Sym}(\omega)$ be the symmetric group of countable degree, let $N = {\mathbb{F}_2}^{\omega}$ be its natural permutation module over $\mathbb{F}_2$, and let M be a countable direct sum of copies of N. The G-action on M is not continuous, but is continuous on each finite direct sum. The problem lies below the group action itself. The real issue is that the inclusion $A \times \oplus B_i \to \oplus(A \times B_i)$ need not be continuous. If $A_1 \supset A_2 \supset \cdots$ is an infinite descending chain of open subsets then the preimage of the open set $\oplus A_i \times B_i$ in $A \times \oplus B_i$ is not open.

By a *weakly profinite G-module M* we intend the following:

(1) M is the countable union of the increasing chain of G-submodules $\{M_i\}_{i<\omega}$.
(2) Each M_i is profinite.
(3) The G-action on each M_i is continuous.
(4) M has the weak topology determined by the M_i.

Again, we refer to the M_i here as a *profinite system* for M.

For much of what follows we could also work with arbitrary directed unions of profinite G-modules (although in a few places would also have to assume that various maps are open, and map bounded sets to bounded sets).

Definition 4.1. Suppose G is a Hausdorff topological group and M is a weakly profinite G-module (with a profinite system $(M_i : i < \omega)$). Define, for $n \geq 0$, the *bounded* cochains (etc.) as follows. Let $C^n_c(G, M)$ be the additive group of continuous bounded maps $G^n \to M$. Note that (as the M_i are submodules) the differential d_n maps $C^n_c(G, M)$ to a subgroup $B^{n+1}_c(G, M)$ of $C^{n+1}_c(G, M)$ and has kernel denoted by $Z^n_c(G, M)$. The bounded cohomology groups $H^n_c(G, M) = Z^n_c(G, M)/B^n_c(G, M)$ are the cohomology groups of the cocomplex formd by these, and the restrictions of the d_n.

Note that if M is profinite (and the profinite system just consists of M), then $H^n_c(G, M) = \tilde{H}^n(G, M)$, as defined previously. Also, the inclusion $M_i \subseteq M_j$ induces a homomorphism $H^n_c(G, M_i) \to H^n_c(G, M_j)$ (as in the long exact sequence).

Lemma 4.2. *With the above notation,* $H^n_c(G, M) \cong \varinjlim H^n_c(G, M_i)$.

PROOF. Clearly $Z_c^n(G,M) = \bigcup_n Z_c^n(G,M_i)$. Consider the natural homomorphism $\phi : Z_c^n(G,M) \to \varinjlim H_c^n(G,M_i)$. This is surjective, and for $f \in Z_c^n(G,M_i)$ we have $\phi(f) = 0$ if and only if $f \in B_c^n(G,M_j)$ for some $M_j \geq M_i$. So the kernel of ϕ is $B_c^n(G,M)$, as required. □

COROLLARY 4.3 (Long exact sequence). *Suppose B is a weakly profinite G-module, A is a closed submodule and $C = B/A$. Then there is a long exact sequence:*

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to H_c^1(G,A) \to \dots$$

$$\dots \to H_c^i(G,C) \to H_c^{i+1}(G,A) \to H_c^{i+1}(G,B) \to H_c^{i+1}(G,C) \to \dots$$

PROOF. Let $\{B_i\}_{i<\omega}$ be a profinite system for B. Write $A_i = A \cap B_i$ and $C_i = B_i + A/A \cong B_i/A_i$ (and note that these give profinite systems for A and C, by Lemma 3.2). Then we have short exact sequences of continuous homomorphisms $0 \to A_i \to B_i \to C_i \to 0$. As the modules are profinite, these maps are open and we have a continuous section of $B_i \to C_i$. So for each i we obtain a corresponding long exact sequence $\mathbf{K}_i$ of continuous cohomology groups, using Lemma 1.1. If $B_i \subseteq B_j$ we also have inclusions $A_i \subseteq A_j$ and $C_i \subseteq C_j$, which induce cocomplex maps $\kappa_{ji} : \mathbf{K}_i \to \mathbf{K}_j$. (Why do squares commute here? All the maps defined are natural, with the possible exception of the connecting maps, which, *a proiri*, depend on the choice of continuous section (but in fact, do not). However one can readily check commutativity of the squares involving these.) Thus one has a directed system of exact cocomplexes, and so we have a exact cocomplex of the direct limits involved. Then Lemma 4.2 gives what we want. □

COROLLARY 4.4. *Suppose M_i are weakly profinite G-modules and $d_i : M_i \to M_{i+1}$ are continuous G-homomorphisms such that*

$$0 \to M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \cdots$$

is exact. Suppose that M_i is acyclic if $i > 0$, that is, $H_c^n(G,M_i) = 0$ for $n > 0$. Then $H_c^*(G,M_0)$ is given by the homology of the fixed point complex

$$M_1^G \xrightarrow{\bar{d}_1} M_2^G \xrightarrow{\bar{d}_2} M_3^G \cdots .$$

Specifically, $H_c^n(G,M_0) \cong \ker\bar{d}_{n+1}/\mathrm{im}\bar{d}_n$.

PROOF. This is a standard induction using the above long exact sequence applied to the short exact sequences

$$0 \to \mathrm{im}d_{m-1} \to M_m \to \mathrm{im}d_m \to 0.$$

Note that $\mathrm{im} d_{m-1} = \mathrm{ker} d_m$ is closed in M_m, and $\bar{d}_{m-1}$ gives an isomorphism between $\mathrm{im} d_{m-1}$ (with the subspace topology) and $M_m/\mathrm{ker} d_m$ (with the quotient topology). □

We conclude this section by showing how weakly profinite modules give acyclic modules and modules with injectivity properties. First we give a fairly general construction of weakly profinite modules.

Suppose G is a permutation group on a countably infinite set. Then G has a countable family $\mathcal{U}$ of open subgroups which forms a base of open neighbourhoods of the identity. Suppose M is a weakly profinite G-module, determined by profinite submodules $\{M_i\}_{i<\omega}$. We define the *coinduced module* $M{\Uparrow}^G$ to be the group of bounded maps $f\colon G \to M$ each of which is equivariant with respect to some open subgroup of G. The G-action is defined by the usual rule: $(gf)(x) = f(xg)$. So $M{\Uparrow}^G$ is the (countable) directed union of modules $M_i{\downarrow}_U^G{\Uparrow}_U^G$ (with the obvious inclusion maps between them), as U ranges over $\mathcal{U}$ and $i < \omega$. As each of these is a profinite G-module we can regard $M{\Uparrow}^G$ as a weakly profinite G-module. Moreover, the topology here does not depend on the choice of $\mathcal{U}$ or the M_i (see Remarks 3.4).

We refer to a countable, discrete, torsion abelian group with trivial G-action as a *torsion* G-module. We can obviously regard this as a weakly profinite as it is a countable union of finite submodules.

Lemma 4.5 (Acyclic modules). *Let $G \in \mathcal{PG}$ with topology generated by a countable family $\mathcal{U}$ of open subgroups, and M a torsion weakly profinite G-module. Then $M{\Uparrow}^G$ is acyclic.*

Proof. By Lemmas 1.4 and 4.2

$$H_c^n(G, M{\Uparrow}^G) = \varinjlim_i \varinjlim_U H_c^n(U, M_i)$$

where U ranges over $\mathcal{U}$ and M_i ranges over the profinite system for M (and by assumption, these are finite). The maps in the inner direct limit are given by restriction. Suppose $f \in Z_c^n(U, M_i)$. We may assume that $f(1, \ldots, 1) = 0$: this holds automatically if n is odd, and if n is even we adjust by the coboundary arising from the constant function with value $f(1, \ldots, 1)$. As f is continuous and $G \in \mathcal{PG}$, there exists an open subgroup $U_1 \leq U$ such that f is zero on U_1^n. The result follows. □

Lemma 4.6 (Injectivity). *Let $G \in \mathcal{PG}$ with topology generated by a countable family $\mathcal{U}$ of open subgroups, and suppose D is a divisible, torsion G-module. Let N be a weakly profinite G-module, let M be a closed submodule of N and suppose $f : M \to D{\Uparrow}^G$ is a continuous*

G-homomorphism. Then there is a continuous G-homomorphism $g : N \to D{\Uparrow}^G$ extending f.

PROOF. Let $\{N_i\}_{i<\omega}$ be a profinite family for N. Set $M_1 = M \cap N_1$. As M is closed in N, this is a compact submodule of N_1. It will suffice to show that $f|_{M_1}$ can be extended to a continuous G-homomorphism $N_1 \to D{\Uparrow}^G$, for then we can proceed inductively (this gives a continuous G-homomorphism with domain $M + N_1$, and the latter is a closed submodule of N).

By Lemma 3.1 there is a finite subgroup F of D and an open subgroup H of G such that $f(M_1) \subseteq F{\Uparrow}_H^G$. Composing this with the evaluation map at 1 we get a continuous H-equivariant homomorphism $f_1 : M_1 \to F$. As D is divisible, this extends to a continuous homomorphism $g_1 : N_1 \to F'$ for some finite subgroup F' of D (this is part of Pontryagin duality for compact abelian groups). By Corollary 2.2 g_1 is H_1-equivariant for some open subgroup H_1 of H. By reciprocity, we obtain a continuous G-homomorphism $g : N \to F'{\Uparrow}_{H_1}^G$ which extends $f|_{M_1}$. □

5. Continuous cohomology on trivial modules

In this section G denotes a permutation group on an infinite set Ω, topologised in the usual way. Although not everything we do requires this, we assume G is *oligomorphic* on Ω: it has finitely many orbits on Ω^n, for all $n \in \mathbb{N}$. By a *continuous* G-space Δ, we mean one in which point stabilisers are open subgroups of G, and on which G has finitely many orbits. In this case, G is oligomorphic on Δ also. Throughout, F denotes a finite abelian group regarded as a trivial G-module. We are interested in computing $H_c^*(G, F)$. We realise this as the homology of a cocomplex constructed from orbits of G on various continuous G-spaces and give a criterion for this cocomplex to be contractible.

We consider F^Ω, the group of functions $f : \Omega \to F$ as a profinite G-module by giving it the product topology and setting $(gf)(x) = f(g^{-1}a)$ (for $g \in G$ and $a \in \Omega$). Note that this is isomorphic to the direct sum of F^{Ω_i} where Ω_i ranges over the G-orbits on Ω. So by the following F^Ω is a direct sum of coinduced modules.

LEMMA 5.1. *If G is transitive on Ω, then $F{\Uparrow}_U^G \cong F^\Omega$ where U is the stabiliser of a point a in Ω.*

PROOF. Well-known: define $\varphi : F^\Omega \to F{\Uparrow}_U^G$ by $\varphi(f)(g) = f(g^{-1}a)$ and check this is an isomorphism of G-modules. □

DEFINITION 5.2. Suppose $(\Delta_i : i \in \mathbb{N})$ is a sequence of continuous G-spaces with surjective G-maps $p^i : \Delta_{i+1} \to \Delta_i$. If $j > i$ define

$p^{j,i} : \Delta_j \to \Delta_i$ to be $p^{j-1} \circ \cdots \circ p^i$. We say that $(\Delta_i, p^i)_{i \in \mathbb{N}}$ is a *full* sequence of G-spaces if for every i there is a transversal T_i of the G-orbits on Δ_i such that for every open $U \subseteq G$ there is $j > i$ such that for every G-orbit X on Δ_j there is $x \in X$ such that $G_x \leq U$ and $p^{j,i}(x) \in T_i$, where G_x denotes the stabiliser of x in G.

Given (Δ_i, p^i) as above we have continuous, injective G-module homomorphisms $q_i : F^{\Delta_i} \to F^{\Delta_{i+1}}$ given by $q_i f = f p^i$. We form the direct limit $\varinjlim F^{\Delta_i}$, and regard this as a weakly profinite G-module.

LEMMA 5.3. *If (Δ_i, p^i) is full, then $\varinjlim F^{\Delta_i}$ is acyclic.*

PROOF. This is similar to the proof of 4.5. Suppose $f : G^n \to F^{\Delta_i}$ is a continuous cocycle. We show that there is $j > i$ such that the image of f in $H^n_c(G, F^{\Delta_j})$ is zero.

Let T_i be the particular system of orbit representatives on Δ_i guaranteed by fullness. Let $s = |T_i|$ and $\bar{f} : G^n \to F^s$ obtained by evaluating at the elements of T_i. Then $\bar{f}$ is continuous and (as in 4.5) we may assume that $\bar{f}(1, \ldots, 1) = 0$. Thus there is an open subgroup U of G such that $\bar{f}$ is identically zero on U^n.

Now let j be as in the definition of fullness. Let $\Delta_{j,k}$ (for $k \leq t$) be the G-orbits on Δ_j. Let $\{x_k : k \leq t\}$ be representatives for these with $p^{j,i}(x_k) \in T_i$ and $U_k = G_{x_k} \leq U$ for all k. Then

$$H^n_c(G, F^{\Delta_j}) \cong \oplus_{k \leq t} H^n_c(G, F^{\Delta_{j,k}}) \cong \oplus_k H^n_c(U_k, F,)$$

the latter from Lemma 5.1 and Shapiro's lemma (1.4). The image of f in the k-th direct factor here is f restricted to U_k^n evaluated at $p^{j,i}(x_k)$ (modulo $B^n_c(U_k, F)$), and this is zero, as required. □

If $i \in \mathbb{N}$, let Ω^i denote the set of i-tuples of elements of Ω, regarded as a (continuous) G-space (with $g(a_1, \ldots, a_i) = (ga_1, \ldots, ga_i)$). Define *simplicial maps* $d_i : F^{\Omega^i} \to F^{\Omega^{i+1}}$ by

$$(d_i f)(a_1, \ldots, a_{i+1}) = \sum_{j \leq i+1} (-1)^j f(a_1, \ldots, \widehat{a_j}, \ldots, a_{i+1}),$$

where as usual the hat means that a_j is omitted from the tuple. Clearly this is a G-map. Then as usual:

LEMMA 5.4. *The sequence*

$$0 \to F \to F^{\Omega} \xrightarrow{d_1} F^{\Omega^2} \xrightarrow{d_2} F^{\Omega^3} \xrightarrow{d_3} \cdots$$

is exact (where F is embedded into F^{Ω} as the constant functions).

COROLLARY 5.5. *Suppose $(\Delta_i, p^i)_{i \in \mathbb{N}}$ is a full sequence of continuous G-spaces. Let $M_{i,j}$ be the G-module $F^{\Delta_i^j}$; let $d_{i,j} : M_{i,j} \to M_{i,j+1}$*

be the simplicial maps; let $q_{i,j} : M_{i,j} \to M_{i+1,j}$ *be induced by the* p^i*. Let* M_j *be the direct limit* $\varinjlim_i M_{i,j}$ *considered as a weakly profinite* G*-module. As* $q_{i,j+1}d_{i,j} = d_{i+1,j}q_{i,j}$*, the* $d_{i,j}$ *induce continuous maps* $d_j : M_j \to M_{j+1}$*. Then:*
(i) the M_j *are acyclic weakly profinite* G*-modules;*
(ii) $0 \to F \to M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3 \cdots$ *is exact;*
(iii)

$$H^n_c(G, F) = \varinjlim \ker(M^G_{i,n+1} \xrightarrow{d_{i,n+1}} M^G_{i,n+2})/\mathrm{im}(M^G_{i,n} \xrightarrow{d_{i,n}} M^G_{i,n+1}).$$

Proof. (i) For fixed j, the sequence of maps $(p^i)^j : \Delta^j_{i+1} \to \Delta^j_i$ is full. So M_j is acyclic, by Lemma 5.3.
(ii) This follows from commutativity of the maps, and exactness of the 'rows' (i.e. for fixed i, from Lemma 5.4).
(iii) This follows from Corollary 4.4. □

Remarks 5.6. Note that $M^G_{i,n+1}$ is finite: it is a direct sum of r copies of F, where r is the number of G-orbits on Δ^{n+1}_i. But in general the groups $H^n_c(G, F)$ need not be finite. Nevertheless, we develop a criterion which in particular cases guaratees that they are zero if $n > 0$.

Definition 5.7. Suppose Δ is a continuous G-space. If $X \subseteq \Delta$, let $G_X = \{g \in G : gx = x \text{ for all } x \in X\}$. Consider a function p which assigns to every finite $X \subseteq \Delta$ a G_X-orbit $p(X)$ on $\Delta \setminus X$. We say that p is a *strong type* (over $\emptyset$) for $(\Delta; G)$ if:
(i) whenever $X_1 \subseteq X_2$ are finite, then $p(X_1) \supseteq p(X_2)$;
(ii) if $g \in G$ and X is finite subset of Δ, then $gp(X) = p(gX)$.

This definition appeared in ([**5**], Definition 2.1), and the terminology originates in model theory. It would be more accurate to refer to p as a type (over Δ) which is non-splitting over the emptyset. In fact, as we are assuming G is oligomorphic, p is a type which is definable over $\emptyset$. In any case, the point of introducing this here is the following.

Lemma 5.8. *Suppose* p *is a strong type for* $(\Delta; G)$*. Then the complex of* G*-fixed points*

$$(F^\Delta)^G \xrightarrow{d_1} (F^{\Delta^2})^G \xrightarrow{d_2} (F^{\Delta^3})^G \xrightarrow{d_3} \cdots$$

is exact.

Proof. We show that the complex is contractible. Note first that $D_i = (F^{\Delta^i})^G$ is the set of maps which are constant on each G-orbit on Δ^i. Define $s_i : D_{i+1} \to D_i$ as follows. Let $f \in D_{i+1}$ and $x \in \Delta^i$. Let $y \in p(x)$. Set $(s_i f)(x) = f(y, x)$. As p is a strong type and f is G-invariant, this is a well-defined function which is G-invariant.

Then one checks that $s_i d_i + d_{i-1}s_{i-1} = -\iota$, the identity map on D_i: the only point at which this deviates from the usual calculation is to notice that if x' is x with one of its coordinates deleted, then $y \in p(x')$. □

From now on, suppose p is a strong type for $(\Omega; G)$. Let Δ_n consist of all $(a_1, \ldots, a_n) \in \Omega^n$ with $a_1 \in p(\emptyset)$ and $a_{i+1} \in p(a_1, \ldots, a_i)$ (for $1 \leq i \leq n-1$): call these *p-sequences* (of length n). Note that G is transitive on Δ_n and $(\Delta_n; G)$ also has a strong type p_n: if X is a finite subset of Δ_n let X_1 be the elements of Ω appearing in tuples in X, and let $p_n(X)$ be the G_X-orbit which contains $(y_1, \ldots, y_n)$ where $y_1 \in p(X_1)$, $y_2 \in p(X_1 \cup \{y_1\})$ .... It is easy to check that p_n is a well-defined strong type on Δ_n. Denote by $\pi^i : \Delta_{i+1} \to \Delta_i$ the map given by projection to the first i coordinates.

Definition 5.9. With the above notation, we say that p is a *full* strong type for $(\Omega; G)$ if for any p-sequence b and finite $X \subseteq \Omega$ there is a p-sequence a extending b such that $G_b \leq G_X$.

Theorem 5.10. *Suppose p is a full strong type for $(\Omega; G)$.*

(i) *The trivial module F is acyclic.*
(ii) *The G-modules F^{Δ_m} are acyclic, for all m.*

Proof. (i) We apply Corollary 5.5 to the sequence of G-spaces $(\Delta_i, \pi^i)_{i \in \mathbb{N}}$. As every open subgroup of G contains some G_X with X finite, and G is transitive on Δ_i, fullness of p means that this is a full sequence. By Lemma 5.8 and the above observations, the groups in the direct limit in Corollary 5.5(iii) are zero if $n > 0$, whence the result.

(ii) Let $c \in \Delta_m$. Define $p'(Y) = p(Yc)$, for Y a finite subset of Ω. This is a full strong type for $(\Omega; G_c)$. So $H^n_c(G_c, F) = 0$ for all $n > 0$. The result then follows from Shapiro's Lemma (1.4) and Lemma 5.1. □

Definition 5.11. Let G be a permutation group. Denote by $\mathcal{M}(G)$ the category whose objects are closed submodules of (profinite) modules $A{\Uparrow}_U^G$, where U is an open subgroup of G and A is a finite abelian group (with trivial G-action). Morphisms are continuous G-homomorphisms.

It is clear that as the modules are compact, the morphisms here are open maps with closed images. Also, $\mathcal{M}(G)$ is an additive category with finite (sums and) products. In general, however, it is not clear that $\mathcal{M}(G)$ will be closed under quotients (by closed submodules).

Corollary 5.12. *Suppose $(\Omega; G)$ is a permutation group with a full strong type p. Suppose also that $\mathcal{M}(G)$ is closed under quotients.*

Then $\mathcal{M}(G)$ is an abelian category with enough acyclics and $H^n_c(G,M)$ is finite for all $n \in \mathbb{N}$ and $M \in \mathcal{M}(G)$.

PROOF. By the above remarks, we clearly have an abelian category. By defintion, any $M \in \mathcal{M}(G)$ embeds into some coinduced module $A{\Uparrow}^G_H$, and any one of these embeds into a finite direct sum of modules F^{Δ_m} (in the notation of Theorem 5.10). But these are acyclic (by Theorem 5.10).

The remaining claim follows by dimension shifting (as in Corollary 4.4) and the fact that $H^0_c(G, F^{\Delta_m}) \cong F$ (because G is transitive on Δ_m). □

COROLLARY 5.13. *Suppose G is one of the following permutation groups. Then $\mathcal{M}(G)$ satisfies the conclusions of Corollary 5.12.*
(i) Sym(Ω) *acting on* Ω.
(ii) An infinite dimensional general linear group over a finite field acting on its natural module.
(iii) The group of isometries of a countable dimensional classical space: a vector space V over a finite field equipped with a symplectic, unitary or non-degenerate quadratic form.

PROOF. In each case the coinduced modules $F{\Uparrow}^G_U$ satisfy the descending chain condition on closed submodules. The same is therefore true for any element of $\mathcal{M}(G)$, and it then follows that $\mathcal{M}(G)$ is closed under quotients. (See 7.2 - 7.7 in [**6**] for this, phrased in rather different language).

So we need to exhibit a full strong type in each case.
(i) Take $p(X) = \Omega \setminus X$.
(ii) Take $p(X)$ to be the vectors independent from X.
(iii) For the orthogonal spaces (not characteristic 2) and for the unitary spaces we take $p(\emptyset) = \{v \in V : (v,v) = 1\}$ and $p(X) = p(\emptyset) \cap (X^\perp \setminus \langle X \rangle)$ (where $(.,.)$ is the bilinear form). Thus p-sequences are orthonormal sequences, and it is clear this gives a full strong type.

For the symplectic spaces we consider instead G acting on Ω, (an orbit on) tuples which enumerate hyperbolic planes (i.e. non-degenerate 2-spaces). This permutation action is faithful and gives rise to the same topology on G as its action on V. If X is a finite subset of Ω let $p(X)$ consist of (suitably enumerated) hyperbolic planes in $\langle X \rangle^\perp$ which are independent from X. By Witt's theorem this is a single G_X-orbit, and p is therefore a strong type. But any finite dimensional subspace is contained in a non-degenerate one, which is the orthogonal sum of hyperbolic planes. So p is full.

A similar argument handles the case of the orthogonal spaces in characteristic 2. □

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