Some remarks on generic structures*

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Abstract

We show that the theory of a generic structure constructed from an amalgamation class given by a predimension (as defined by Hrushovski) can be undecidable and have the strict order property. This answers a question of Pourmahdian. By contrast, if the generic is $\aleph_0$-categorical, then we show that it satisfies Shelah's property $NSOP_4$, but need not satisfy $NSOP_3$.

Introduction

In this note we collect together some observations about generic structures constructed using Hrushovski's method of predimensions. We shall particularly be concerned with where the theories of these can fit in the hierarchy:

$$\text{simple} \Rightarrow NSOP_3 \Rightarrow NSOP_4 \ldots \Rightarrow NSOP.$$

Here $NSOP$ is the negation of the strict order property and $NSOP_n$ is Shelah's strengthening of it from [9] (we repeat the definition in Section 2).

Before describing the results, we recall briefly some details of the construction method. The original version of this is in [4], where it is used to provide a counterexample to Lachlan's conjecture, and [5], where it is used to construct a non-modular, supersimple $\aleph_0$-categorical structure. The book [11] is a very convenient reference for this (see Section 6.2.1). Generalisations and reworkings of the method (particularly relating to simple theories) are also to be found in [2], [7], [8].

We work with a relational language $L = \{R_i : i \in I\}$ with finitely many relations of each arity. Suppose $\overline{\mathcal{K}}$ is a univeral class of L-structures which is

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closed under *free amalgamation*: if $B, C \in \overline{\mathcal{K}}$ have a common substructure, then their disjoint union $B \otimes_A C$ over A is also in $\overline{\mathcal{K}}$. Suppose further that the R_i are realised by tuples of distinct elements in structures in $\overline{\mathcal{K}}$. Denote by $\mathcal{K}$ the finite structures in $\overline{\mathcal{K}}$.

Now let $(\alpha_i : i \in I)$ be a sequence of non-negative real numbers. Define $d_0(A) = |A| - \sum_i \alpha_i |R_i[A]|$, for $A \in \mathcal{K}$. If $A \subseteq B \in \mathcal{K}$ write $A \leq B$ to mean $d_0(A) < d_0(B')$ for all $A \subset B' \subseteq B$. (Remark: one sometimes says that 'A is self-sufficient in B'. This is denoted by $\leq^*$ in Pourmahdian's papers). For structures in $\mathcal{K}$, one has:

- If $X \subseteq B$ and $A \leq B$, then $X \cap A \leq X$;
- If $A \leq B \leq C$, then $A \leq C$.

Consequently, for each $B \in \mathcal{K}$ there is a closure operation given by $\mathrm{cl}_B(X) = \bigcap\{A : A \leq B, X \subseteq A\}$ for $X \subseteq B$.

The relation $\leq$ can be extended to infinite structures so that the above properties still hold. If $M \in \overline{\mathcal{K}}$ and $A \subseteq M$, write $A \leq M$ to mean that $A \cap X \leq X$ for all finite $X \subseteq M$.

Now define $\mathcal{K}_0$ to be $\{A \in \mathcal{K} : \emptyset \leq A\}$, and similarly $\overline{\mathcal{K}}_0$. Then $(\mathcal{K}_0, \leq)$ and $(\overline{\mathcal{K}}_0, \leq)$ satisfy a strong form of the amalgamation property over $\leq$-substructures (see 6.2.9 of [11], for example):

- If $A_1, A_2 \in \overline{\mathcal{K}}_0$ have a common substructure A_0 and $A_0 \leq A_1$, then $A_2 \leq A_1 \otimes_{A_0} A_2 \in \overline{\mathcal{K}}_0$.

It follows that there is a countable structure $M_0 \in \overline{\mathcal{K}}_0$ which is the union of a chain of finite self-sufficient substructures and satisfies the $\leq$-homogeneity condition:

- If $A \leq M_0$ is finite and $A \leq B \in \mathcal{K}_0$, there is an embedding of B over A into M_0 whose image is self-sufficient in M_0.

Equivalently, any $B \in \mathcal{K}_0$ is isomorphic to a self-sufficient substructure of M_0, and isomorphisms between finite self-sufficient substructures of M_0 extend to automorphisms of M_0.

The structure M_0 is unique up to isomorphism and is called the *generic structure* associated to the amalgamation class $(\mathcal{K}_0, \leq)$ (see [6]).

The closure in the generic is locally finite but not uniformly so, so in other models of $Th(M_0)$ one can have the closure of some finite set being infinite. Indeed, (for example, if one of the α_i is rational and sufficiently small) cl need not be contained in algebraic closure. In Section 1 we look at a particular example where this is the case and show that $Th(M_0)$ is undecidable and has the strict order property. This answers Question 4.10 in [8] (and contradicts remarks in Section 4.2 of [5]).

In Section 2 we look at a variation of the construction (also from [5]) where closure is locally finite. For this, we have a continuous, increasing $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $f(x) \to \infty$ as $x \to \infty$ and we consider $\mathcal{K}_f = \{A \in \mathcal{K}_0 : d_0(X) \geq f(|X|) \; \forall X \subseteq A\}$. For suitable choice of f (- call these *good* f), $(\mathcal{K}_f, \leq)$ is closed under free amalgamation over $\leq$-substructures and we have an associated generic structure M_f. In this, the closure is uniformly locally finite, and so M_f is $\aleph_0$-categorical. In ([5], Section 4.3), Hrushovski gave an example where M_f is supersimple of SU-rank 1: the point is to choose f carefully so that one has the independence theorem holding over closed sets (the argument is also given in ([11], 6.2.27) and in more generality in ([2], Theorem 3.6)). Here we show that if f is good, then M_f is $NSOP_4$. Finally, we give an example where M_f has SOP_3.

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1 $Th(M_0)$ is bad

To keep the ideas clear, we shall work with a particular example.

The language L has (apart from equality) a single ternary relation R. $\overline{\mathcal{K}}$ consists of L-structures in which this is symmetric, and if $A \in \mathcal{K}$, then $d_0(A) = |A| - |R[A]|$, where by $R[A]$ we mean the set of 3-subsets of A picked out by R (rather than the set of 3-tuples). Note that if $a, b, c \in A \in \overline{\mathcal{K}}$ and $A \models R(a, b, c)$, then $c \in \mathrm{cl}_A(a, b)$.

The idea is to encode graphs into the closures of pairs of elements. A similar (but more difficult) type of encoding is used in Section 3 of [10].

Define predicates V, E as follows: $V(x; y, z) \leftrightarrow R(x, y, z)$ and $E(x_1, x_2; y, z) \leftrightarrow V(x_1; y, z) \wedge V(x_2; y, z) \wedge (\exists w) R(x_1, x_2, w)$. If $a, b \in A \in \overline{\mathcal{K}}_0$ then $\Gamma(a, b, A)$ is the graph with vertex set $V[A; a, b]$ and edges $E[A; a, b]$. Note that the vertex set here is in $\mathrm{cl}_A(a, b)$ and any edge is witnessed in $\mathrm{cl}_A(a, b)$. Thus if $A \leq B$ then $\Gamma(a, b, A) = \Gamma(a, b, B)$.

Now, if Γ is any graph, there is $A_\Gamma \in \overline{\mathcal{K}}_0$ and $a, b \in A_\Gamma$ with $\Gamma(a_\Gamma, b_\Gamma, A_\Gamma) = \Gamma$. Indeed, suppose Γ has vertex set S and edge set $U \subseteq [S]^2$. Let A_Γ be the disjoint union of $\{a_\Gamma, b_\Gamma\}$, S and U with the relation $R[A_\Gamma]$ given by $\{a_\Gamma, b_\Gamma, s\}$ for all $s \in S$, and $\{s_1, s_2, u\}$ for all $u = \{s_1, s_2\} \in U$. It is easy to check that $A_\Gamma \in \overline{\mathcal{K}}_0$ and all points in A_Γ are in the closure of a_Γ, b_Γ.

Given any first-order sentence σ in the language of graphs (with binary relation S) we construct an L-formula $\theta_\sigma(y,z)$ by replacing all atomic subformulas $S(x_1,x_2)$ in σ by $E(x_1,x_2;y,z)$ and replacing any quantifier $\forall x$ by $\forall x \in V(x;y,z)$ (and likewise $\exists x$ by $\exists x \in V(x;y,z)$).

Lemma 1.1 *For any $M \in \overline{\mathcal{K}}_0$ and $a,b \in M$ we have:*

$$\Gamma(a,b,M) \models \sigma \Leftrightarrow M \models \theta_\sigma(a,b).$$

Proof. This is essentially a triviality: cf. Theorem 5.3.2 in [3]. $\square$

Now let M_0 be the generic structure for the class $(\mathcal{K}_0,\leq)$, as in the introduction.

Theorem 1.2 *Suppose σ is a sentence in the language of graphs. Then there is a finite model of σ iff $M_0 \models (\exists y,z)\theta_\sigma(y,z)$.*

Proof. If there is a finite model Γ of σ then we can find $A \leq M_0$ with $A_\Gamma \cong A$. Then by the lemma, $M_0 \models \theta_\sigma(a_\Gamma,b_\Gamma)$, as required.

Conversely suppose $a,b \in M_0$ and $M_0 \models \theta_\sigma(a,b)$. Then $\Gamma(a,b,M_0)$ is a graph which is a model of σ. It is finite, as it is contained in $\mathrm{cl}_{M_0}(a,b)$. $\square$

Corollary 1.3 *$Th(M_0)$ is undecidable.*

Proof. The construction of θ_σ from σ is obviously recursive. On the other hand, the theory of all finite graphs is undecidable (by Trakhtenbrot's Theorem). So the same is true of $Th(M_0)$, by the above. $\square$

Theorem 1.4 *Suppose σ is a sentence in the language of graphs which has arbitrarily large finite models. Then some infinite model of σ is interpretable in a model of $Th(M_0)$.*

Proof. The formulas $\theta_\sigma(a,b) \wedge$ '$|V(x;a,b)| \geq n$' (for $n \in \mathbb{N}$) are consistent with $Th(M_0)$ by assumption and compactness. So there is a model M of $Th(M_0)$ and $a,b \in M$ such that $\Gamma(a,b,M)$ is an infinite model of σ (by the Lemma). $\square$

Corollary 1.5 *$Th(M_0)$ has the strict order property.*

Proof. We can construct a family of finite graphs in which arbitrarily large finite linear orderings are uniformly interpretable. There is a sentence in the language of graphs which implies that the interpreted stucture is a linear ordering (- again, this is by Theorem 5.3.2 of [3]). Thus, arguing by compactness as in the previous proof, there is a model M of $Th(M_0)$ and $a, b \in M$ such that the interpreted structure in $\Gamma(a, b, M)$ is an infinite linear ordering. But $\Gamma(a, b, M)$ is itself interpreted in M. $\square$

The reader will have noticed that the proofs used only the local finiteness of closure and $\leq$-universality of M_0 (i.e. every $A \in \mathcal{K}_0$ is isomorphic to some self-sufficient substructure of M_0).

The undecidability result means that $Th(M_0)$ is not recursively axiomatisable. In particular, the *semigeneric theory* T_{sgen} given in ([8], Definition 3.27) following [1], does not axiomatize $Th(M_0)$ (for the notation there, we take T_0 as the universal theory describing $\overline{\mathcal{K}}_0$). We have $T_{sgen} \subseteq Th(M_0)$ (essentially, because of the strong form of the amalgamation property), so we conclude that T_{sgen} is not complete. In fact, it is useful to see this in a different way. It is fairly easy to show that if $A \in \overline{\mathcal{K}}$ then there is a model M of T_{sgen} which has A as a self-sufficient substructure. Let σ be some formula in the language of graphs which has only infinite models, let Γ be such a model and $A = A_\Gamma$. Then $M \models (\exists y, z)\theta_\sigma(y, z)$, but of course $M_0 \not\models (\exists y, z)\theta_\sigma(y, z)$.

The undecidability and SOP rested on transferring properties of finite structures to M_0. So one possible positive property left for $Th(M_0)$ is the following:

Question 1.6 Does $Th(M_0)$ have the finite model property?

2 Strong order properties

2.1 M_f has $NSOP_4$

Recall the following from ([9], Definition 2.5).

Definition 2.1 Suppose T is a complete first-order theory and $n \geq 3$ is an integer. Say that T has the property SOP_n (-strong order property n) if there exists a formula $\phi(\bar{x}, \bar{y})$ and a sequence of tuples $(\bar{a}_i : i < \omega)$ in some model N of T such that
(a) $N \models \phi(\bar{a}_i, \bar{a}_j)$ for $i < j < \omega$;

(b) $M \models \neg\exists \bar{x}_0 \ldots \bar{x}_{n-1} \phi(\bar{x}_0, \bar{x}_1) \wedge \phi(\bar{x}_1, \bar{x}_2) \wedge \cdots \wedge \phi(\bar{x}_{n-1}, \bar{x}_0)$.
The negation of this property is denoted by $NSOP_n$.

Allowing the formula to have parameters changes nothing. Also, we may take the sequence $(\bar{a}_i : i < \omega)$ to be indiscernible (over whatever parameters). Condition (b) simply says that there are no directed n-cycles in the directed graph determined by the relation $\phi(\bar{x}, \bar{y})$.

It is conjectured that SOP_4 is a 'good dividing line' for existence of universal models, so it seems worthwhile to show that various structures are $NSOP_4$ (- so, in principle, fall on the right side of the dividing line). Recall the notation of the introduction. We have a continuous, increasing $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ with $f(x) \to \infty$ as $x \to \infty$ and we consider $\mathcal{K}_f = \{A \in \mathcal{K}_0 : d_0(X) \geq f(|X|)\ \forall X \subseteq A\}$. We *assume* that $(\mathcal{K}_f, \leq)$ is closed under free amalgamation: if $A \leq B_1, B_2 \in \mathcal{K}_f$ then the disjoint union of B_1 and B_2 is in $\mathcal{K}_f$. (This is a condition on the growth of f.) In particular, there is a generic model M_f for $\mathcal{K}_f$, and this is $\aleph_0$-categorical.

Theorem 2.2 *With the above notation, $Th(M_f)$ has the property $NSOP_4$. In particular, M_f does not have the strict order property.*

Proof. Work in a big model N of $Th(M_f)$ and suppose $(a_i : i < \omega)$ is an infinite indiscernible sequence of tuples in N (over a finite parameter set, which we may assume to be $\emptyset$). Let $p(x_0, x_1)$ be the complete type of (a_0, a_1) in N. To show that $Th(M_f)$ is $NSOP_4$ it will be enough to show that

$$p(x_0, x_1) \cup p(x_1, x_2) \cup p(x_2, x_3) \cup p(x_3, x_0)$$

is consistent.

We now follow the notation and some of the arguments from [2] very closely. In particular if B is a finite subset of N then $d(B)$ is the minimum value of $d_0(B_1)$ when $B \subseteq B_1 \subset N$. (This minimum is attained on some finite set of size $\leq f^{-1}(d_0(B))$). The structure M_f is the special case $y(B) = |B|$ of the examples in ([2], Section 3). The conditions on f in ([2], 3.1) are irrelevant by our current assumptions on f, so ([2], Theorem 3.6(i)) holds, and (M_f, d_0) has properties (P1-P4, P6, P7) of [2]. The notation $d(c/S)$ is defined at the start of Section 2.5 (and on p. 259) of [2] and acl denotes algebraic closure in N.

Claim: There is a finite set c of parameters such that $(a_i : i < \omega)$ is c-indiscernible and for $i = 1, 2$ we have $d(a_i/ca_0 \ldots a_{i-1}) = d(a_i/c)$ (i.e. a_0, a_1, a_2 are d-independent over c).

The proof is as in paragraphs 2 and 3 of the proof of 2.19(b) in [2], but we repeat the outline here. Extend the indiscernible sequence to an indiscernible sequence $(a_i : i \in \mathbb{Z})$. Let $A_0 = \mathrm{acl}(a_i : i < 0)$. Then $(a_i : i \geq 0)$ is A_0-indiscernible and d-independent over A_0. By extending the sequence, and then thinning, we may assume that $X = \mathrm{acl}(A_0 a_{i_2}) \cap \mathrm{acl}(A_0 a_{i_0} a_{i_1})$ is constant for $i_0 < i_1 < i_2$ and then that $(a_i : i \in \omega)$ is X-indiscernible. By (P7) there is a finite $C \subseteq X$ such that $d(a_2/a_0 a_1 C) = d(a_2/C)$, and C-indiscernibility gives the d-independence of a_0, a_1, a_2. ($\Box$ Claim)

Note that (as M_f is a generic) $\mathrm{tp}(a_i, a_j/c)$ is determined by the isomorphism type of $E_{ij} = \mathrm{cl}(a_i a_j)$. Let $C = \mathrm{cl}(c)$, let $E_i = \mathrm{cl}(a_i c)$ and let $A = E_{01} \cup E_{12}$. So by the d-independence of a_0, a_1, a_2 over c we have that A is the free amalgam of E_{01} and E_{12} over E_1. Moreover $E_0 \cup E_2 = A \cap E_{02} \leq A$ and $E_0 \cup E_2$ is the free amalgam of E_0 and E_2 over C. By the latter, there is an isomorphism $\gamma : E_0 \cup E_2 \to E_0 \cup E_2$ over C which interchanges the tuples a_0, a_2.

Consider the embeddings $h_1 : E_0 \cup E_2 \to E_{02}$ given by inclusion and $h_2 : E_0 \cup E_2 \to E_{02}$ given by appling γ and then inclusion. Let F be the free amalgam obtained from these embeddings and $g_i : E_{02} \to F$ such that $g_1 \circ h_1 = g_2 \circ h_2$. By assumption $F \in \mathcal{K}_f$, so we can assume that (an isomorphic copy of) $F \leq N$. Let $a'_0 = g_1(h_1(a_0)), a'_1 = g_1(a_1), a'_2 = g_1(h_1(a_2))$ and $a'_3 = g_2(a_1)$. Then

$$\mathrm{tp}(a_0, a_1) = \mathrm{tp}(a'_0, a'_1) = \mathrm{tp}(a'_1, a'_2) = \mathrm{tp}(a'_2, a'_3) = \mathrm{tp}(a'_3 a'_0)$$

as required. To see that the types are equal, one simply has to consider closures of the two tuples (- they are even equal over the image of c in F). $\Box$

It seems entirely plausible that this argument could be extended to cover the more general examples of M_f given in Section 3 of [2] which are obtained by 'iterating' the predimension construction (and even starting with a vector space, rather than a pure set). Some care has to be taken formulating the hypothesis concerning free amalgamation (- essentially it should hold at all levels of the iteration), but we see no real problem with this.

2.2 A structure with SOP_3

It is well-known that if f is chosen appropriately, then the structure M_f is simple. Somewhat surprisingly though, there does not appear to be an example in the literature where a structure M_f is shown not to be simple.

At the risk of filling a much needed gap, we give an example here. In fact, we construct an M_f with the property SOP_3 (- so it is not simple, by ([9], Claim 2.7)). We leave the reader to judge whether the delicacy of the argument justifies or mitigates against the inclusion of this example (- we tend towards the former).

We work with graphs and use $\sim$ to denote adjaceny. For a finite graph A we let $d_0(A) = 2|A| - e(A)$, where $e(A)$ denotes the number of edges of A and let $y(A) = 2|A|$. The function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is piecewise linear and satisfies the following properties:

F1 $f(3n) \leq f(n) + 1$ for $n \geq 10$;

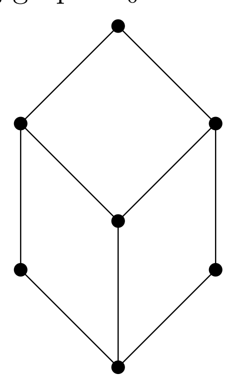
F2 $f'(y) \leq 1/y$ for $y \geq 14$;

F3 $f(y) \to \infty$ as $y \to \infty$;

F4 $f(0) = 0$; $f(2) = 2$; f(4)=3; $f(8) = 4$; $f(14) = 5\frac{1}{6}$; $f(18) \leq 6$.

F5 $f(\frac{1}{2}k^2) \leq k$ if $k \geq 6$.

For convenience (and to fit with usage in [2]) we let $\mathcal{K}_f$ consist of finite graphs B such that for all subgraphs A we have $f(y(A)) \geq d_0(A)$. Thus the class $\mathcal{K}_f$ omits the 3-cycle and includes 4-cycles. The value of $f(14)$ is chosen so the $\mathcal{K}_f$ omits the following graph X_0.



Thus $\mathcal{K}_f$ fails to have property $P5$ of [2] (called ITD in [7]).

Claim 1. $(\mathcal{K}_f, \leq)$ has the free amalgamation property.

Lemma 3.3 and Remarks 3.8 in [2] show that because of property (F2), we only have to check this with structures having fewer than 7 vertices. We omit the details. □ Claim 1)

We now work in the the situation of the proof of Theorem 3.6(ii) of [2]: we are looking for the precise instances of where the axiom P5 (equivalently, the ITD) fails. Thus we have a finite graph E having subgraphs E_0 and E_1, E_2, E_3 which are d-independent (in E) over E_0. Let E_{ij} denote the $\leq$-closure in E of E_i and E_j, and assume that E is the union of these, and that $E_{ij} \cap E_{jk}$ is the free amalgam of E_{ij}, E_{jk} over E_j. We assume that each E_{ij} is in $\mathcal{K}_f$. The question is to determine whether $E \in \mathcal{K}_f$. Because of X_0, we know that this is not always so. However, the following shows that this is essentially the only example, for the particular f.

Claim 2. Suppose $D \subseteq E$ and let $D_{ij} = D \cap E_{ij}$ and $D_i = D \cap E_i$. If $d_0(D) < f(y(D))$, then each D_{ij} has at most 4 points. In fact, D must be isomorphic to X_0.

Let $d_{ij} = d_0(D_{ij})$ and without loss assume that d_{12} is the largest of these. Let $D' = D_{12} \cup D_{13}$. The amalgamation lemma means that $D' \neq D$. Moreover, Claim 2 on p. 175 of [2] gives that

$$D' \leq^* D. \tag{1}$$

Also,

$$d_0(D') = d_{12} + d_{13} - d_1 \geq d_{12} + 1 \tag{2}$$

as $D_1 \leq D_{13}$ (- because $E_1 \leq E_{13}$).

Let $f^{-1} = g$. Suppose we have $g(d_{12}+1) \geq 3g(d_{12})$. Then

$$y(D) \leq \sum_{i<j} y(D_{ij}) \leq 3g(d_{12}) \leq g(d_{12}+1).$$

By Equation (2), this is at most $g(d_0(D'))$ and by Equation (1) this is $\leq g(d_0(D))$. Thus $d_0(D) \geq f(y(D))$.

Now, if $g(x) \geq 10$, then $f(3g(x)) \leq f(g(x)) + 1$ by condition (F1), and so $3g(x) \leq g(f(g(x))+1) \leq g(x)+1$. Thus if $d_0(D) < f(y(D))$, then $g(d_{12}) < 10$, so $d_{12} < f(10) = 4\frac{7}{18}$. So $d_{ij} < 4\frac{7}{18}$, i.e. $y(D_{ij}) < 10$. So $|D_{ij}| \leq 4$, as required. The final part, identifying D as X_0 is simply a matter of considering small cases (the condition $f(18) \leq 6$ in (F4) is needed to exclude a 9-vertex graph as another possibility for D). ($\square$ Claim 2)

Now let $r \in \mathbb{N}$ and $A = \{a_i : i \leq r\}$, $B = \{b_i : i \leq r\}$. Define a graph $\Gamma(A, B, x_0)$ with vertext set $\{x_0\} \cup A \cup B \cup \{z_{ij} : i < j \leq r\}$ and adjacencies:

$$x_0 \sim a_\ell, b_\ell;\ z_{ij} \sim a_i, b_j$$

for $i < j$, and no others.

Claim 3. $\Gamma(A, B, x_0) \in \mathcal{K}_f$.

Suppose $X \subseteq \Gamma = \Gamma(A, B, x_0)$. We must show that $d_0(X) \geq f(2|X|)$. Clearly we may assume $X \leq \Gamma$, so $x_0 \in X$ and if $a_i, b_j \in X$ (with $i < j$), then $z_{ij} \in X$. Similarly, if $z_{ij} \in X$, then $a_i, b_j \in X$. Let $X_A = X \cap A, X_B = X \cap B$ and $m = |X_A| + |X_B|$. Then

$$d_0(X) = d_0(x_0 \cup X_A \cup X_B) = m + 2 =: k$$

(using our assumptions on X).

Moreover $\frac{1}{2}y(X) = |X| \leq 1+m+|X_A||X_B|$. This is at most $1+m+\frac{1}{4}m^2 = (1 + \frac{1}{2}m)^2 = \frac{1}{4}k^2$. Hence $y(X) \leq \frac{1}{2}k^2$. So $d_0(X) \geq f(y(X))$ provided that $f(\frac{1}{2}k^2) \leq k$. If $k \geq 6$ then property (F5) gives this. The small graphs X with $k \leq 5$ can easily be check to lie in $\mathcal{K}_f$. ($\Box$ Claim 3)

Claim 4. $\{x_0\} \cup A \leq \Gamma(A, B, x_0)$, and A, B are d-independent over x_0 in $\Gamma(A, B, x_0)$.

The first part follows from the above calculation of $d_0(X)$. For the second, it is clear that $x_0 \cup A$ and $x_0 \cup B$ are freely amalgamated over x_0 and $x_0 \cup A \cup B \leq^* \Gamma(A, B, x_0)$. So the claim follows from 2.3 of [2].

Now let $C = \{c_i : i \leq r\}$ and consider

$$E^r = \Gamma(A, B, x_0) \cup \Gamma(B, C, x_0) \cup \Gamma(C, A, x_0).$$

So a_i, b_j have a common neighbour $z_{ij} \neq x_0$ iff $i < j$; b_j, c_k have a common neighbour $z'_{jk} \neq x_0$ iff $j < k$; and c_k, a_i have a common neighbour $z''_{ki} \neq x_0$ iff $k < i$.

Claim 5. $E^r \in \mathcal{K}_f$.

It is easy to check that E^r has no subgraph isomorphic to X_0, so the claim follows from Claim 2.

Now, $E^r \leq E^{r+1}$, so we may assume that the E^r form an increasing chain of $\leq$-substructures of M_f. Let $\bar{a}_i = (a_i, b_i, c_i)$ (for $i < \omega$) and let $\phi(x_1, x_2, x_3, y_1, y_2, y_3)$ be the formula with parameter x_0 which says:

- $x_0 \sim x_i, y_i$ for $i = 1, 2, 3$
- $(\exists z, z', z'')(z \sim x_1, y_2) \wedge (z' \sim x_2, y_3) \wedge (z'' \sim x_3, y_1)$

Then $M_f \models \phi(\bar{a}_i, \bar{a}_j)$ whenever $i < j$. Moreover, as M_f has no subgraph isomorphic to X_0, the relation determined by ϕ on M_f has no (directed) 3-cycle. Thus we have proved:

Theorem 2.3 *If f satisfies conditions F1-F5, then M_f has property SOP_3.* $\square$

We omit the verification that there is a function f satisfying our conditions F1-F5: essentially all one has to do is make f grow like $\log_3$ (- see 5.1 of [2] for this sort of thing).

Remarks 2.4 It might be interesting to have an example of M_f which is not simple and is $NSOP_3$. Perhaps this can be done as above, but with an inductive construction of f which alternately controls the failure of the ITD (as in Claim 2) and blocks the creation of an indiscernible sequence (by using f to exclude certain subgraphs).

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