Trivializing the Hrushovski constructions

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UEA, February 2004.

EHUD HRUSHOVSKI: (1988) Counterexamples to two of the most significant conjectures in model theory.

QUESTION: Are the counterexamples just very clever pathologies, or do they have connections with other parts of mathematics?

THIS TALK:

- Model-theoretic background
- Zilber's conjecture
- Hrushovski constuctions
- Random graphs (Shelah, Spencer; Baldwin)
- New way of looking at the constructions (DE)

1. Model theory

The *formulas* of a first-order language L are certain finite strings of the symbols:

(1)

$$\forall \; \exists \; \neg \;\; \rightarrow \;\; \wedge \; \vee \;) \; (\; , x_1 \, x_2 \; \ldots \;\; y_1 \, y_2 \; \ldots$$

and

(2) Various symbols (incuding $=$) used to denote relations and functions.

What you take for (2) depends on what sort of structure you want the formulas to talk about.

EXAMPLES : (i) Graphs: $=$ and a 2-ary relation R for adjacency.

(ii) Rings: $=$ and $+$, $\cdot$ (2-ary functions), 0, 1 (constants).

(iii) K-vector spaces: $=$, $+$, 0, and for each $\alpha \in K$ a 1-ary function symbol to denote scalar multiplication by α.

L-FORMULAS: Usual mathematical shorthand: variables can only range over the *elements* of a structure.

NOTATION:(i) $M \models \phi$ the formula ϕ is true in the structure M.

(ii) If $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a formula with free variables amongst $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $\bar{a} = (a_1, \ldots, a_m) \in M^m$, let

$$\phi[M, \bar{a}] = \{(b_1, \ldots, b_n) \in M^n : M \models \phi(b_1, \ldots, b_n, \bar{a})\}$$

This is a *definable subset of* M^n (using *parameters* $a_1, \ldots, a_m$).

GENERAL PHILOSOPHY: Fix a language L and:

(I) Compare L-structures by looking at their L-*theories*

$$Th(M) = \{\phi : \phi \text{ closed and } M \models \phi\}.$$

(II) For a given L-structure M, think about its collection of definable subsets.

EXAMPLES FOR (I): What properties can be expressed by first-order formulas?

Graphs:

- Triangle free (YES)
- Diameter $\leq d$ (YES)
- Connected (NO)

Rings:

- Integral domain (YES)
- Bézout (YES)
- Principal ideal domain (NO)

2. Zilber's Conjecture.

DEFINITION: An infinite L-structure M is *strongly minimal* if for every L-formula $\phi(x, \bar{y})$ there exists $k \in \mathbb{N}$ such that for all $\bar{a}$, either $\{b \in M : M \models \phi(b, \bar{a})\}$ or its complement has size $\leq k$.

From the viewpoint of (II), these are the 'simplest' structures.

EXAMPLES OF STRONGLY MINIMAL STRUCTURES:
(1) M is a 'pure set' (the language L has $=$, but no other relation or function symbols).
(2) M is a K-vector space (where K is a division ring and the language is as described before).
(3) M is an algebraically closed field (the language is the language for rings).

ZILBER'S CONJECTURE: These are essentially the only examples of strongly minimal structures.

Early 1980's. THEOREM (Zilber *et al.*): The conjecture is true for ω-categorical structures.

1988. Without any further hypotheses, the conjecture is false (Hrushovski).

Early 1990's. Under additional hypotheses (Zariski structure) the conjecture is true (Hrushovski, Zilber).

1990's - date. New idea of Zilber: Realise the counterexamples in 'classical' mathematics using complex analytic functions.
Work of Zilber, Wilkie, Koiran, Peatfield....

2003. Zilber: Connections between the construction and non-commutative geometry, string theory...

3. The construction

Describe the simplest form of the construction.

Work with graphs (so L has $=$ and a 2-ary relation symbol R).

Fix a real parameter α with $0 < \alpha < 1$.

DEFINITION:

(1) If A is a finite graph define the *predimension* of A to be

$$\delta(A) = |A| - \alpha e(A)$$

where e denotes the number of edges in A.

(2) If A is a subgraph of the finite graph B write

$$A \leq B$$

to mean

$$\delta(A) \leq \delta(B') \text{ for all } B' \text{ with } A \subseteq B' \subseteq B.$$

(Pronounced: A is a self-sufficient subgraph of B.)

PROPERTIES:

(1) If $A \leq B$ and $X \subseteq B$, then $A \cap X \leq X$.

(2) If $A \leq B \leq C$, then $A \leq C$.

(3) If $A_1, A_2 \leq B$, then $A_1 \cap A_2 \leq B$.

(4) If $X \subseteq B$, there is a unique smallest $A \leq B$ with $X \subseteq A$. Call this the *closure* of X in B, and denote it by $\mathrm{cl}_B(X)$.

Denote by $\mathcal{C}$ the class of finite graphs A which satisfy

$$\emptyset \leq A$$

i.e. for all $X \subseteq A$, we have $|X| - \alpha e(A) \geq 0$. (Another way: average valency of X is $\leq 2/\alpha$.)

STRONG AMALGAMATION LEMMA: Suppose $B, C \in \mathcal{C}$ and A is a subgraph of both B and C, and $A \leq C$. Let E be the disjoint union of B and C over A. Then $E \in \mathcal{C}$ and $B \leq E$.

Using this, we can 'glue' the graphs in $\mathcal{C}$ together to obtain:

THEOREM: There exists a countably infinite graph $M = M_\alpha$ satisfying the following properties:

(G1): M is the union of a chain of finite subgraphs $A_1 \leq A_2 \leq A_3 \leq \cdots$ all in $\mathcal{C}$.

(G2): If $A \leq M$ is finite and $A \leq B \in \mathcal{C}$, then there is an embedding $f : B \to M$ which is the identity on A and has $f(B) \leq M$.

Moreover, M is uniquely determined up to isomorphism by these two properties and if $h : B_1 \to B_2$ is an isomorphism between finite closed subgraphs of M, then h can be extended to an automorphism of M. □

THEOREM: (Hrushovski; Wagner; Baldwin, Shi) If $0 < \alpha < 1$ then M_α is stable (and not 1-based). If α is rational, then M_α is ω-stable, of infinite Morley rank. □

4. Irrational α, random graphs

S. Shelah, J. Spencer, (JAMS, 1988): Fix α **irrational** with $0 < \alpha < 1$. For $n \in \mathbb{N}$, consider choosing a graph on n vertices by randomly choosing each pair of vertices to be an edge, with probability $1/n^{\alpha}$. If ϕ is a closed L-formula, let

$$P(\phi, \alpha; n)$$

be the probability that the randomly chosen graph has the property expressed by ϕ. Consider what happens as $n \to \infty$:

THEOREM: (Zero-one law) For each such ϕ, either
$P(\phi, \alpha; n) \to 0$ as $n \to \infty$, or
$P(\phi, \alpha; n) \to 1$ as $n \to \infty$. □

Later on, Baldwin and Shelah made the connection:

THEOREM: For all closed L-formulas ϕ:

$$P(\phi, \alpha; n) \to 1 \text{ as } n \to \infty \Leftrightarrow M_\alpha \models \phi.$$

REMARKS: (1) Compare with the classic result of Fagin, Glebskii *et al.*. If we choose the edges with probability $\frac{1}{2}$, then we again have a zero-one law, but this time the limit theory is that of the Random Graph.

(2) If β is **rational** and $0 < \beta < 1$ then as $\alpha \to \beta^-$ (and α irrational), then $Th(M_\alpha) \to Th(M_\beta)$.

5. α rational; directed graphs

DIRECTED GRAPHS: Let $\mathcal{D}$ be the class of finite **directed** graphs D with all vertices having ≤ 2 out-vertices. If $C \subseteq D$, write $C \sqsubseteq D$ to mean that out-vertices of elements of C are contained in C (say that C is closed in D).

EASY LEMMA: (1) If $C \sqsubseteq D$ and $X \subseteq D$ then $C \cap X \sqsubseteq X$.
(2) If $C \sqsubseteq D \sqsubseteq E$ then $C \sqsubseteq E$.
(3) (Strong Amalgamation) Suppose $D, E \in \mathcal{D}$ and C is a sub-digraph of both D and E and $C \sqsubseteq E$. Let F be the disjoint union of D and E over C. Then $F \in \mathcal{D}$ and $D \sqsubseteq F$. $\square$

Using this we have:

PROPOSITION: There exists a countably infinite digraph N satisfying the following properties:
(D1): N is the union of a chain of finite subgraphs
$C_1 \sqsubseteq C_2 \sqsubseteq C_3 \sqsubseteq \cdots$ all in $\mathcal{D}$.
(D2): If $C \sqsubseteq N$ is finite and $C \sqsubseteq D \in \mathcal{D}$, then there is an embedding $f : D \to N$ which is the identity on C and has $f(D) \sqsubseteq N$.

Moreover, N is uniquely determined up to isomorphism by these two properties and is $\sqsubseteq$-homogeneous. $\square$

PROPOSITION: N is stable, trivial and 1-based. $\square$

... So N is rather a dull structure.

.... or is it?

Fix $\alpha = \frac{1}{2}$. Work with $\delta(A) = 2|A| - e(A)$.
So $\mathcal{C} = \{A : \delta(X) \geq 0$ for all $X \subseteq A\}$ and $M = M_{1/2}$.

THEOREM: Forget the directions on the edges in N. The resulting graph is $M_{1/2}$.

The following answers a question of Bruno Poizat from 1991.

COROLLARY: There is a stable, trivial, 1-based structure with a reduct which is neither trivial, nor 1-based.

DEFINITION: Suppose A is a finite graph. A $\mathcal{D}$-*orientation* of A is a directed graph $A^+ \in \mathcal{D}$ with the same vertex set as A and such that if we forget the direction on the edges, we obtain A.

The theorem is a fairly straightforward corollary of the following two lemmas:

LEMMA 1: (1) Suppose B is a finite graph. Then

$$B \in \mathcal{C} \Leftrightarrow B \text{ has a } \mathcal{D}\text{-orientation.}$$

(2) If $B \in \mathcal{C}$ and $A \subseteq B$, then $A \leq B$ iff there is a $\mathcal{D}$-orientation of B in which A is closed.

LEMMA 2: If $A \leq B \in \mathcal{C}$ then any $\mathcal{D}$-orientation of A extends to a $\mathcal{D}$-orientation of B. □