Report on EPSRC Grants GR/R32048/01 and GR/R32123/01

Model theory of generic structures and simple theories

As intended in the original proposal, the grants employed Dr Massoud Pourmahdian as a research assistant at UEA for a year, starting in October 2001. Pourmahdian was then employed by Edinburgh for a year, starting in October 2002. The original grant applications were submitted as being related and this report covers activities supported by both of the grants. The start dates were later than in the original proposals due to Pourmahdian's commitments in Iran. Pourmahdian spent much of the second year based in Oxford, where Macintyre was visiting on sabbatical leave from Edinburgh.

A. Description of Reseach Undertaken

The objectives were to study open problems in model theory by investigating and extending Hrushovski's constructions of generic structures. The work done focused around the following themes:

(1) Hrushovski's unstable, non uniformly locally finite generic.
(2) Stable forking and Lascar strong types in generics.
(3) Construction of non-CM-trivial structures.
(4) Model theory of complex exponentiation.

These correspond to 1, 2, 4 and 5 in the original objectives, with some variations which we shall explain below. We made no progress on one of the original objectives: the possibility of using a generic construction to build a new o-minimal structure.

1. Hrushovski's unstable, non uniformly locally finite generic.

In [13], Hrushovski considers a relational language L and a linear predimension δ defined on finite L-structures. Denote by K_0 those finite L-structures for which δ is non-zero on all non-empty substructures, and if A is a substructure of $B \in K_0$, then write $A \leq^* B$ to mean that $\delta(B') > \delta(A)$ for all $A \subset B' \subseteq B$. Then $\langle K_0; \leq^* \rangle$ is a (free) amalgamation class and we denote by M_0 the countable generic structure for this. There is a natural extension by definitions L_+ of L and an inductive L_+-theory T_{Nat} associated to $\langle K_0; \leq^* \rangle$ ([13]).

In [13] it is claimed that T_{Nat} has a model companion (i.e. there is an axiomatization of the class $EX(T_{Nat})$ of existentially closed models of T_{Nat}) and that $Th(M_0)$ is simple. The first of these claims was refuted by Pourmahdian in [16]. When we wrote the original proposal, Pourmahdian had nevertheless shown that $EX(T_{Nat})$ was simple in the sense of [14]. He has subsequently improved this and shown:

(1.1) Theorem [17] *$EX(T_{Nat})$ is a simple Robinson theory (in the sense of* [13]*).*

When we wrote the proposal we envisaged that we would be able to axiomatise $Th(M_0)$ and show it to be simple. In fact the opposite is the case (and strongly refutes Hrushovski's original claims):

(1.2) Theorem [7] *$Th(M_0)$ is undecidable (so has no recursive axiomatization) and has the strict order property (so is not simple).*

This can be seen as a rather negative result about $Th(M_0)$. However, it raises a number of interesting problems, which Evans and Pourmahdian looked at, but made only partial progress on:

(1.3) Questions: *Which structures can appear as $\leq^*$-closures of finite sets in models of $Th(M_0)$? Does $Th(M_0)$ have the finite model property? In a sufficiently saturated model of $Th(M_0)$, is the type of a tuple determined by the isomorphism type of its $\leq^*$-closure?*

In [1], Baldwin and Shi introduce the notion of a *smooth class* $\langle K; \sqsubseteq \rangle$ as a way of axiomatising some of the Hrushovski constructions. This is extended further in [18], and used to prove simplicity of the generic structure for certain smooth classes *with AC*: essentially where algebraic closure in a saturated elementary extension of the generic coincides with the closure associated to $\sqsubseteq$. The example $\langle K_0; \leq^* \rangle$ is an example of a smooth class without AC and (1.2) above shows that the results of [18] cannot be extended in general. In view of (1.2), the following result of Pourmahdian is somewhat surprising:

(1.4) Theorem [20] *There is a smooth class without AC where the theory of the generic model is recursively axiomatizable and simple.*

2. Stable forking and Lascar strong types in generics.

The stable forking conjecture asks whether, in a simple theory, if a complete type forks over a particular subset of its domain, then there is an instance of a stable formula in the type which forks over the subset. Both Evans [5] and Pourmahdian [16, 19] had verified the stable forking conjecture for some of the simple theories produced using the Hrushovski constructions. The original intention of the proposal was to combine these approaches and prove sfc for a wider class of generics. However, it became clear that there might be a possibility to produce a counterexample to the sfc using predimensions and generic structures and Pourmahdian pursued this (obviously more interesting) direction. At present, he only has a counterexample in the generalized model theory sense of [3] (– so not as a full first-order theory). Specifically:

(2.1) Theorem [21] *There exists a simple homogeneous model without the stable forking property.*

The approach is interesting in its own right as it makes use of a predimension taking values in a non-standard extension of the reals.

One of the most important open questions about simple theories concerns Lascar strong types. In a monster model, the equivalence relation of having the same Lascar strong type (over the empty set) is the transitive closure of the relation of having the same type over some (variable) submodel. The question is whether, in a simple theory, this is the same equivalence relation as having the same strong type (over the empty set). Briefly, we say that in this case *Lstp* = *stp*. The extent to which Lascar strong type refines strong type is measured (for each strong type) by a compact, connected group (called the compact Lascar group). An example of a simple (rank 1) Robinson theory having a strong type with a non-trivial compact Lascar group is given in [13].

Our original proposal did not specifically refer to plans to work on this problem, but the approach to (2.1) suggested various ideas about how to construct a simple theory in which *Lstp* $\neq$ *stp*. These were vigorously pursued by Pourmahdian, in collaboration with Wagner (in Lyon). There are still significant obstacles to this approach and the examples with a non-trivial Lascar group constructed so far by these ideas are (as with Hrushovski's result), only simple as Robinson theories.

3. Construction of non-CM-trivial structures.

In the stable and simple structures produced by the Hrushovski constructions which do not explicitly involve an infinite field, the independence relation of non-forking satisfies a property called *CM-triviality* which restricts its complexity (cf. [12], [5]). So our original proposal was to try to vary the constructions in order to obtain non-CM-trivial stable or simple structures (not interpreting an infinite field). At the time, the only such structure which was known was the free pseudospace in [2]. Although this is ω-stable and non-CM-trivial, non-forking is *trivial* (in the sense of [10]: pairwise independence of a set of tuples implies independence), so is, in a very strong sense, degenerate. One of the main results of [6] is:

(3.1) Theorem [6] *There exists a non-trivial, non-CM-trivial stable structure which does not interpret an infinite field. The non-CM-triviality can be witnessed by elements all of the same type.*

One can destroy the triviality (and preserve the non-CM-triviality) in the Baudisch-Pillay example very easily by taking the disjoint union of it with, say, a vector space. But really this is avoiding the issue and the last clause of (3.1) indicates that something more significant is happening here.

In [15], Pillay introduces the notion of *n-ampleness* giving a hierarcy of complexity of non-forking generalizing the idea of CM-triviality. So, 1-ample is the same as being non-1-based; 2-ample is the same as being non-CM-trivial, and a stable infinite field is n-ample for all $n \in \mathbb{N}$. It is far from clear how to extend the construction in [2] in order to obtain n-ample structures (for $n \geq 3$). However, the construction in (3.1) generalizes in a very natural fashion and one obtains:

(3.2) Theorem [6] *For every $n \in \mathbb{N}$ there exists a stable, n-ample structure which does not interpret an infinite field.*

The structures in (3.1) and (3.2) are stable, but not superstable. The ultimate goal would be to construct, say, strongly minimal sets which are n-ample (and do not interpret an infinite field). Also of interest here would be ω-categorical examples, either stable, or (super)simple. At present we have no idea how this might be achieved.

The structures in (3.1), (3.2) are constructed as reducts of trivial stable structures. Thus an unexpected bonus is the following, which answers a question from [10]:

(3.3) Theorem [6] *There is a trivial stable theory with a non-trivial reduct.*

Perhaps the most surprising aspect of these constructions via reducts is that one can also obtain some of the Hrushovski examples in this way. The most basic version of the Hrushovski construction is sometimes called the *ab initio* example: it produces ω-stable structures of infinite rank. One has:

(3.4) Theorem [8] *The ab initio Hrushovski structures which are ω-stable of infinite rank are reducts of stable, trivial, 1-based structures.*

There are obstructions to proving the same sort of result for the strongly minimal sets from [12]. Nevertheless the viewpoint of [8] makes the constructions seem, in some sense, much more natural and simplifies considerably the proof of the main amalgamation lemma in [12]. The paper [9] applies these ideas to give a solution to a problem posed by P. Cameron and C. Praeger.

The results mentioned in this section are due to Evans, but discussions with Pourmahdian were invaluable in obtaining these results.

4. Model theory of complex exponentiation.

Macintyre had considerable experience with logical aspects of Schanuel's Conjecture, both on universal - algebraic aspects, and on its key role in the decidability of the real exponential field. In particular, he had been involved much earlier in the unconditional construction of exponential fields satisfying Schanuel's Conjecture. Zilber's later constructions using predimension are much deeper, but much of the algebraic machinery is similar (the difference being the delicate model theory of predimension).

After orientation by Pourmahdian, Macintyre began an analysis designed to show Zilber's main conjecture is false, i.e. to show that the complex exponential field is *not* isomorphic to Zilber's 'categorical' exponential field of cardinal continuum. The issue is to show that Zilber's elementary criteria for solvability of exponential equations fail in $\mathbb{C}$. That some non-trivial instances hold is verified by classical methods from the theory of entire functions. Detailed study of this led to a constructive analysis of the distribution of complex zeros of exponential polynomials, and classical results about the zeros clustering along certain *lines*. This gives a workable criterion in $\mathbb{C}$ for when certain systems have *no* solutions. It is to be stressed that the constraint comes from the topology of $\mathbb{C}$, not from merely the exponential algebra. It appears that some of the unsolvable systems have solutions in Zilber's model, but some delicate exponential algebra has to be checked. It is expected that the work will be completed soon (the analytic part is complete, and of independent interest).

In the course of the analysis, Macintyre became more keenly aware of the issue of identifying a (the?) real analogue of Zilber's model. He has some ideas on how to force pathological models of Zilber type with no large real exponential subfield. Note that one does not know intelligable axioms for the real exponential, though it is decidable. While Zilber's exponential is undecidable, it has clear axioms.

B. Dissemination and Further Collaboration.

Most of the papers resulting from activity funded by the grants are still only available in the form of preprints. Those of Evans can be obtained from his webpage (www.uea.ac.uk/~h120/publications.html). Those not already submitted for publication will be submitted to international journals in the next few months.

All three investigators attended the international meeting on Simple Theories ('Simpleton-02') at CIRM, Luminy, in 2002. Pourmahdian gave a 1-hour talk on Theorem 2.1; Evans gave a half-hour talk on a preliminary version of Theorem 3.2. Pourmahdian also gave talks on the material in Section A2 in Leeds and Oxford. He made several trips to Lyon to discuss this material with Wagner. His discussions with Lessmann (EPSRC-funded postdoc in Oxford) were also very useful. Evans has also given seminars in Leeds and Oxford about results in Section A3. He gave a 1-hour invited talk on Hrushovski constructions (including Theorems 1.2 and 3.3) at the EURESCO conference on Algebra and Discrete Mathematics in Hattingen, 2003.

Part of the grant money at Edinburgh was used to fund a collaboration between de Piro (a Research Fellow at Edinburgh, working with Macintyre) and Kim (MIT) on the group configuration in simple theories. The paper [4] is a result of this collaboration. Tomasic (an EPSRC funded postdoc at Leeds) constructed a new Weil cohomology theory along lines opened up by Macintyre, and is using Weil cohomology in the study of definable sets. Macintyre hopes to involve de Piro in a related project.

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