Finite covers with finite kernels

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Abstract

We are concerned with the following problem. Suppose Γ and Σ are closed permutation groups on infinite sets C and W and $\rho : \Gamma \longrightarrow \Sigma$ is a non-split, continuous epimorphism with finite kernel. Describe (for fixed Σ) the possibilities for ρ. Here, we consider the case where ρ arises from a finite cover $\pi : C \longrightarrow W$. We give reasonably general conditions on the permutation structure $\langle W; \Sigma \rangle$ which allow us to prove that these covers arise in two possible ways. The first way, reminiscent of covers of topological spaces, is as a covering of some Σ-invariant digraph on W. The second construction is less easy to describe, but produces the most familiar of these types of covers: a vector space covering its projective space.

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0 Introduction

Various natural model-theoretic questions can be cast into the following form:

> Given a structure, what additional structure can be put on one part of it without affecting the structure on a different part?

One such question which has received a certain amount of attention recently, largely as a result of work on totally categorical structures, is the problem of determining the *finite covers* of a given first-order structure W (see, for example, [1, 2, 4, 5, 9, 10, 12, 13]). Loosely speaking, a finite cover C of W is obtained in two steps. First, one replaces each element of W by a copy of some finite structure, to obtain a *free* cover C_0. So W can be identified with a subset of C_0^{eq}. Then C is obtained from C_0 by adding extra relations on C_0 in such a way that no new definable relations are induced on the copy of W.

The structures W considered in the papers cited above are all countable and $\aleph_0$-categorical. In these circumstances one tends to think (and work) in terms of permutation groups: if C is a finite cover of a countable, $\aleph_0$-categorical (or just ω-saturated) structure W then there is a natural epimorphism $\rho : Aut(C) \longrightarrow Aut(W)$, between the automorphism groups. In this paper we study finite covers where the kernel of this epimorphism is finite, and (to avoid trivialites) the epimorphism is non-split.

From a model-theoretic viewpoint, this special case of the problem plays an important role in the analysis of an arbitrary finite cover (see the main results, 2.1 and 4.7, of [5]). It is also the heart of the general problem in the case where W has trivial algebraic closure (cf. the reductions in [4]).

From a group-theoretic point of view, it is not too fanciful to compare this problem with the determination of the Schur multiplier of a group (although as this is usually defined in terms of generators and relations (see [11]), one cannot import this terminology with any degree of accuracy). Indeed, our results can be used to prove statements which are phrased solely in group-theoretic terminology: for example (Theorem 2.13)

Theorem 0.1 *Suppose Γ is a perfect group having a finite, central subgroup Z such that Γ/Z is isomorphic to the symmetric group on a countably infinite set. If Γ has a subgroup of index $< 2^{\aleph_0}$ which is disjoint from Z, then $Z = 1$.* $\square$

Before describing our main results, we give some precise definitions.

0.1 Permutation structures

We shall phrase most of our results in permutation group-theoretic language, using the following terminology.

Definition 0.2 (1) If X is any set, we denote by $Sym(X)$ the symmetric group on X, that is, the group of all permutations of X. As usual, we regard $Sym(X)$ as a topological group with pointwise stabilisers of finite subsets of X forming a base of open neighbourhoods of the identity. A subgroup of $Sym(X)$ is closed if and only if it is the automorphism group of some first-order structure with domain X.

(2) A pair $\langle W; G\rangle$ is a *permutation structure* if W is a non-empty set, and G is a closed subgroup of $Sym(W)$. We refer to W as the *domain* of the permutation structure. The elements of G are called *automorphisms* of the permutation structure, and G itself is the *automorphism group* of the permutation structure. We generally denote the automorphism group by $Aut(W)$, and remorselessly abuse notation by referring to the permutation structure simply as W. So when we do this it is implicit that some closed permutation group $Aut(W)$ is also given. We say that two permutation structures are *biinterpretable* if their automorphism groups are isomorphic as topological groups.

(3) Suppose W is a permutation structure. If X is a subset of W and Y a subset of W, or more generally of some set on which $Aut(W)$ is acting (usually in some natural way), then $Aut(X/Y)$ is the group of permutations of X which extend to automorphisms of W fixing every element of Y. We occasionally use an alternative notation for this. If G is a group acting on a set X and Y is a subset of X, then $G_Y = \{g \in G : gy = y \text{ for all } y \in Y\}$.

(4) Suppose W is a permutation structure. Note that a subgroup of finite index in $Aut(W)$ is closed if and only if it is open. Now, we say that a permutation structure W is *irreducible* if $Aut(W)$ has no proper closed subgroup of finite index. If X is a finite set, then $Aut^0(W/X)$ is the intersection of the closed subgroups of finite index in $Aut(W/X)$.

(5) Let W be a permutation structure and X a finite subset of W. Let $n \in \mathbf{N}$. Then an *n-type* over X is an $Aut(W/X)$-orbit on W^n. If P is such a thing and $x \in W^n$ then we write $P(x)$ to indicate that $x \in P$. We say that an n-type P over X is *stationary* if for every closed subgroup H of finite index in $Aut(W/X)$, H is transitive on P.

In applications, our permutation structures $\langle W; G\rangle$ are obtained as follows. W is countable and the domain of a strongly ω-homogeneous first-order structure $\mathbf{W}$ (for example, $\mathbf{W}$ is countable and ω-saturated, or even $\aleph_0$-categorical), and G is the automorphism group of $\mathbf{W}$. In this case a type in the permutation structure is just the set of realisations in $\mathbf{W}$ of a type in the usual model-theoretic sense. If W is $\aleph_0$-categorical the condition that W be irreducible is equivalent to the condition that $acl(\emptyset) = dcl(\emptyset)$ in $\mathbf{W}^{\mathbf{eq}}$. Furthermore, if $\mathbf{W}$ is also stable, then our usage of the word 'stationary' coincides with its normal usage in stability theory, by virtue of the finite equivalence relation theorem of Shelah.

0.2 Finite covers

Definition 0.3 (1) Suppose C, W are permutation structures. We say that a surjection $\pi : C \longrightarrow W$ is a *finite cover* of W if

(i) the fibres $\{\pi^{-1}(w) : w \in W\}$ of π are finite, and form an $Aut(C)$-invariant partition of C;

(ii) the induced map $\rho : Aut(C) \longrightarrow Sym(W)$ given by

$$\rho g(w) = \pi(g\pi^{-1}(w))$$

has image $Aut(W)$.

We shall call the map ρ *restriction*, and whenever we have a finite cover $\pi : C \longrightarrow W$, and refer to $Aut(C)$ acting on W, it will always be assumed implicitly that the action is via this map.

(2) We frequently use the notation $C(w)$ to denote the fibre $\pi^{-1}(w)$ above w in a cover $\pi : C \longrightarrow W$. If $\pi : C \longrightarrow W$ is a finite cover, the *fibre group* on $C(w)$ is $Aut(C(w))$. The *binding group* on $C(w)$ is $Aut(C(w)/W)$. This is a

normal subgroup of the fibre group. If $Aut(C(w)) = Aut(C(w)/W)$ for every $w \in W$, the finite cover is said to be *untwisted.*

(3) Suppose $\pi : C \longrightarrow W$ is a finite cover. We refer to $Aut(C/W)$ as the *kernel* of the cover (or sometimes we say the *kernel* of π) and say that the cover is *superlinked* if $Aut(C/W)$ is finite, and *trivial* if it is the trivial group. For distinct $w, x \in W$, we say that the cover is *locally trivial* with respect to (x, w) if $Aut(C(x)/C(w)) = 1$. It is *locally trivial* if it is locally trivial with respect to some (x, w). The cover *splits* if there is a closed complement to the kernel in $Aut(C)$.

(4) The finite cover $\pi : C \longrightarrow W$ is *transitive* if $Aut(C)$ is transitive on C (in which case, $Aut(W)$ is also transitive on W) and for $w, x \in W$ we say that the cover is *locally transitive* (with respect to (x, w)) if $Aut(C(x)/C(w))$ is transitive on $C(x)$.

Note that whether or not a finite cover is locally trivial (or locally transitive) with respect to a particular (w, x) depends only on the type R of (w, x) over $\emptyset$. So in this case we refer to the cover being R-locally trivial (or R-locally transitive).

0.3 Outline of results

Our main objective in this paper is to attack the following, which appeared as a question at the end of [4], and which we now state as a conjecture. Note that by the *rank* of a finite abelian group, we mean the smallest number of elements needed to generate it.

Conjecture Suppose W is a countable, irreducible $\aleph_0$-categorical structure which is *G-finite*, that is, for every finite subset X of W, $Aut^0(W/X)$ is of finite index in $Aut(W/X)$. Then there is a natural number r such that the kernel of any irreducible, superlinked finite cover of W has rank at most r.

Thus, we are interested in developing techniques which are as general as possible, but are nevertheless effective computationally. Most of the results are phrased in terms of permutation structures.

We study countable, transitive, irreducible superlinked finite covers. First, we make some easy observations which break this problem down somewhat. Every transitive, superlinked finite cover of a permutation structure W can be written as a composition of a trivial cover of W and an untwisted cover (see Lemma 1.6). We make no further comments about trivial covers, and from now on all finite covers considered will be *untwisted.* By Lemma 3.1, such a cover can be written as a composition of a cover which is locally trivial (with respect to some given $x, w \in W$) and a cover which is locally transitive (with respect to some 2-type related to (x, w)) , and so we consider each of these types of covers separately.

Locally trivial covers of a transitive, irreducible, permutation structure L are analysed in Section 1 under the assumption that there is a *graphic triple of types* on L. The reader should look at Section 1.3 for the definition of this, but in particular it implies that there is some 2-type R on L (over $\emptyset$) such that the digraph $(L; R)$ is connected. The main result of Section 1 (Theorem 1.13) is that under these circumstances, any untwisted, R-locally trivial finite cover of L arises from a *covering* of the digraph $(L; R)$, as defined in [4] (the definitions are repeated here in Section 1.2). We showed in [4] that any such covering arises as a homomorphic image of a *universal covering*, described in terms of homotopy classes of paths in $(L; R)$. Thus we can modify a standard argument from algebraic topology (– the one which says that the first homology group of a path-connected space is the abelianisation of its fundamental group) to produce an abelian group $H(L)$, described as a section of the permutation module $\mathbf{Z}R$, which has as its finite homomorphic images all kernels of irreducible, locally trivial, finite covers of L (Theorem 1.17). Using this, we can show easily that if L is $\aleph_0$-categorical then there is a bound on the rank of such a kernel (Corollary 2.4). This provides a partial answer to the above conjecture.

Unfortunately, we have no general results which guarantee the existence of graphic triples (even if L is $\aleph_0$-categorical), although for many structures they are not hard to find (see Section 2 for examples and further discussion). In Section 2.1 we introduce the notion of a *strong type* on L. If L is irreducible and has a strong type (for example, the latter is guaranteed if L is saturated and stable, in the model-theoretic sense) then there is a graphic triple of types, and indeed (for the appropriate 2-type R) the universal covering of $(L; R)$ is isomorphic to $(L; R)$, and so any irreducible, R-locally trivial finite cover of L is actually trivial (Corollary 1.18).

In Section 3, we consider irreducible, untwisted, locally transitive, superlinked finite covers of a permutation structure W. By Lemma 3.3, the kernel of such a cover is a homomorphic image of the stabiliser of two points in W, and there is a necessary condition on the stabiliser of three points of W for such a cover to exist. In particular, this provides a better answer to the conjecture: if W has a graphic triple of types, and we only consider untwisted covers, then the conjecture is true.

Rather remarkably, we show in Theorem 3.5 that if W arises from a countable, irreducible $\aleph_0$-categorical structure which has a strong type (for example, a structure which is $\aleph_0$-categorical and stable) then the necessary conditions in Lemma 3.3 on the kernel are actually sufficient (this is the only point in the paper where we use some traditional model theory: towards the end of the proof we take an elementary extension of W). Thus in this case we have a complete description of the irreducible, superlinked finite covers of W. The construction in Theorem 3.5 produces the archetypal example of a non-trivial, irreducible superlinked finite cover: a vector space covering its projective space.

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1 Covers and Coverings

In a previous paper ([4], Section 5) we showed how coverings of digraphs could be used to give examples of non-split superlinked finite covers of the universal, homogeneous, countable local order. The aim of this section is to prove a rather general converse to this. We will show (Theorem 1.13) that the existence of a certain sort of 3-type on a transitive, irreducible structure L implies that, for certain distinct $x, w \in W$, any irreducible finite cover $\tau : C \longrightarrow L$ with $Aut(C(x)/C(w)) = 1$ arises from a covering of digraphs.

We first prove some easy lemmas on finite covers, and then review the material we need from [4] on coverings of digraphs.

1.1 Finite covers

The first part of the following lemma (which, for countable C, is essentially (1.2) of [5]) shows that the condition in (1ii) of Definition 0.3 that ρ has image $Aut(W)$ can be weakened to the image of ρ being dense in $Aut(W)$.

Lemma 1.1 *(i) Suppose C is a countable permutation structure and $\pi : C \longrightarrow W$ is a finite-to-one surjection whose fibres form an $Aut(C)$-invariant partition of C. Then the restriction map $\rho : Aut(C) \longrightarrow Sym(W)$ maps closed subgroups of $Aut(C)$ to closed subgroups of $Sym(W)$.*

(ii) Suppose C, W are permutation structures, and $\pi : C \longrightarrow W$ is a finite cover. Then the restriction map ρ sends any open subgroup of $Aut(C)$ to an open subgroup of $Aut(W)$.

Proof. (i) It is enough to show that the image of ρ is a closed subgroup of $Sym(W)$. So suppose $g \in Sym(W)$ is in the closure of the image of ρ. Thus, for every finite subset X of W there exists $h \in Aut(C)$ with $\rho(h)|X = g|X$.

If A, B are finite subsets of $C \cup W$, then an *elementary* map $f : A \longrightarrow B$, is one such that there exists $h \in Aut(C)$ such that $f(x) = h(x)$ for $x \in A \cap C$ and $f(x) = \rho(h)(x)$ for $x \in A \cap W$. For $X \subseteq W$ let $C(X)$ be the union of the fibres $\pi^{-1}(x)$ for $x \in X$.

Now, for each finite $X \subseteq W$, look at the elementary maps $f : C(X) \longrightarrow C(gX)$ (there are finitely many of these). Discard any such f for which there is a finite subset Y of W containing X such that $f \cup (g|Y)$ is not elementary. Let $M(X)$ be the set of those which remain. We claim that this is non-empty. Indeed, suppose it were. List the elementary maps $C(X) \longrightarrow C(gX)$ as $f_1, \ldots, f_n$

and let Y_i be finite, with $X \subseteq Y_i \subseteq W$, and such that $f_i \cup (g|Y_i)$ is not elementary. Let $Y = \bigcup_{i \leq n} Y_i$. Then this is finite, and $g|Y$ is not elementary, which is a contradiction.

Now note that if X, Y are finite subsets of W and $X \subseteq Y$, then ρ_{YX} given by restriction from $C(Y)$ to $C(X)$, gives a map from $M(Y)$ to $M(X)$. For if $f \in M(Y)$ and Z is a finite subset of W containing X, then $f \cup g|(Z \cup Y)$ is elementary, and therefore $f|C(X) \cup g|Z$ is elementary.

Thus we have a directed system of non-empty finite sets $\rho_{YX} : M(Y) \longrightarrow M(X)$ for X, Y finite subsets of W and $X \subseteq Y$, and by Lemma 1.2 of [6] the inverse limit of this system is non-empty. But any element of the inverse limit gives an automorphism h of C with $\rho(h) = g$. Thus g is in the image of ρ, as required. $\Box$ (i).

(ii) Let X be a finite subset of W. It is enough to prove that $\rho(Aut(C/C(X)))$ is open. So we must find a finite subset Y of W, containing X, such that any automorphism of W which fixes all elements of Y extends to an automorphism of C fixing all elements of $C(X)$. As $C(X)$ is finite, we can choose a finite $Y \subseteq W$ such that $Aut(C(X)/Y)$ is as small as possible. Then $Aut(C(X)/Y) = Aut(C(X)/W)$. Let $g \in Aut(W/Y)$. Extend g to some $g_1 \in Aut(C/Y)$. There exists $k \in Aut(C/W)$ with $g_1|C(X) = k|C(X)$. Then $k^{-1}g_1 \in Aut(C/C(X))$ extends g, as required. $\Box$

We also note the following easy results.

Lemma 1.2 *Suppose $\pi : C \longrightarrow W$ is an irreducible, superlinked finite cover. Then*

(i) $Aut(C/W)$ is central in $Aut(C)$;

(ii) the cover is split if and only if it is trivial.

Proof. (i) As $Aut(C/W)$ is a finite normal subgroup of $Aut(C)$ its centraliser in $Aut(C)$ is a closed subgroup of finite index in $Aut(C)$. Irreducibility of $Aut(C)$ now shows that $Aut(C/W)$ is central in $Aut(C)$.

(ii) This follows at once from the assumption that the kernel of the cover is finite. $\Box$

Lemma 1.3 *Suppose $\pi : C \longrightarrow W$ is a transitive finite cover whose kernel is central in $Aut(C)$. Then $Aut(C/c, W) = 1$ for any $c \in C$.*

Proof. Suppose $h \in Aut(C/W)$ fixes $c \in C$. Let $g \in Aut(C)$. Then $h(gc) = g(hc) = gc$. So as $Aut(C)$ is transitive on C, h fixes every element of C. $\Box$

The second part of the next lemma gives a useful condition on W which guarantees that covers are untwisted (this was used extensively in [4]).

Lemma 1.4 *Let $\pi : C \longrightarrow W$ be a transitive, superlinked, irreducible finite cover.*

(1) Suppose that π is untwisted. Then restriction $Aut(C/C(w)) \longrightarrow Aut(W/w)$ is an isomorphism.

(2) Suppose that $Aut(W/w)$ is irreducible. Then for any $x \in W$, the restriction map $j : Aut(C/W) \longrightarrow Aut(C(x))$ is an isomorphism, so the cover is untwisted.

Proof. (1) By Lemmas 1.2 and 1.3 this map is injective. Let $g \in Aut(W/w)$. As π is a finite cover, there exists $g' \in Aut(C)$ such that $g' \mid W = g$. So g' stabilises $C(w)$, whence there exists $h \in Aut(C/W)$ such that $h \mid C(w) = g' \mid C(w)$. Let $g'' = h^{-1}g'$. Then $g'' \in Aut(C/C(w))$ and $g'' \mid W = g$.

(2) By Lemma 1.3, j is injective. So let $g \in Aut(C/x)$. Restriction maps $Aut(C/C(x))$ to a closed subgroup of finite index in $Aut(W/x)$, by Lemma 1.1, and by irreducibilty this must equal $Aut(W/x)$. So there exists $h \in Aut(C/C(x))$ such that $\rho g = \rho h$. Then $h^{-1}g \in Aut(C/W)$ and $h^{-1}g \mid C(x) = g \mid C(x)$. Thus j is surjective. □

The next lemma gives a condition on W which guarantees that superlinked covers are locally trivial.

Lemma 1.5 *Suppose $\pi : C \longrightarrow W$ is a transitive finite cover whose kernel is central in $Aut(C)$. Suppose for some $x, w \in W$ that $Aut(W/x, w)$ is irreducible. Then $Aut(C(x)/C(w)) = 1$.*

Proof. Let $\rho : Aut(C) \longrightarrow Aut(W)$ be the restriction map. By Lemma 1.1 and the irreducibility assumption, $\rho(Aut(C/x, w)) = Aut(W/x, w)$ and furthermore, $\rho(Aut(C/C(x), C(w)))$ is a closed subgroup of finite index in this, so, by hypothesis, must equal $Aut(W/x, w)$ also. Now let $g \in Aut(C/x, C(w))$. There exists $g' \in Aut(C/C(x), C(w))$ such that $\rho(g) = \rho(g')$. Then $g'^{-1}g \in Aut(C/C(w), W)$ and so by Lemma 1.3 is equal to the identity. In particular, $g \mid C(x) = 1$. So $Aut(C(x)/C(w)) = 1$, as required. □

Finally, we have a lemma which factors out the twisted part of a superlinked finite cover as a trivial cover.

Lemma 1.6 *Let $\pi : C \longrightarrow W$ be a transitive, irreducible superlinked finite cover. Then there exist finite covers*

$$\pi_1 : C \longrightarrow C' \text{ and } \pi_2 : C' \longrightarrow W$$

such that π_2 is trivial, π_1 is untwisted and $\pi = \pi_2\pi_1$.

Proof. This follows from Lemma 1.8 of [5]. Let $K = Aut(C/W)$ and let C' be the set of K orbits on C. By 1.1, the group of permutations induced by

$Aut(C)$ on C' is closed, and we take this to be $Aut(C')$. Set $\pi_1(c) = Kc$ and $\pi_2(Kc) = \pi c$, for $c \in C$. Then $Aut(C')$ is naturally isomorphic to $Aut(C)/K$, so $Aut(C'/W)$ is trivial. Moreover, the fibre group in π_1 is isomorphic to K, which is also $Aut(C/C')$. □

1.2 Coverings of digraphs

Definition 1.7 (1) If L is a set, a *digraph* on L is an irreΩexive binary relation $R \subseteq L^2$ which is either symmetric or anti-symmetric. In this case, we refer to the digraph as $(L;R)$. Elements of L will sometimes be called *vertices*, and elements of R *edges* of $(L;R)$. We frequently supress reference to the relation R if it is clear from the context, and refer to the digraph simply as L. If $(L;R)$ is a digraph and $a \in L$ we let

$$a^+ = \{a' \in L : R(a,a')\}$$

and

$$a^- = \{a' \in L : R(a',a)\}.$$

(2) A *path* in the digraph $(L;R)$ is a sequence $x_0x_1\ldots x_n$ of vertices such that for each $0 \leq i \leq n-1$, either $R(x_i, x_{i+1})$ or $R(x_{i+1}, x_i)$. The digraph is *connected* if for all $x, y \in L$ there exists a path starting at x and ending at y.

(3) Let $(L;R)$ be a connected digraph. Let $\mathcal{S}$ be the set of paths in L. We say that $p_1, p_2 \in \mathcal{S}$ are *elementarily homotopic* if one can be obtained from the other by one of the following operations (*elementary homotopies*):

- replace a subsequence of consecutive terms abc where $R(a,b)$, $R(b,c)$, $R(a,c)$ by ac;
- replace a subsequence of consecutive terms abc where $R(b,a)$, $R(c,b)$, $R(c,a)$ by ac;
- replace a subsequence of consecutive terms aba by a.

We say that p_1, p_2 are *homotopic* if one can be obtained from the other by a sequence of elementary homotopies. This is clearly an equivalence relation on $\mathcal{S}$, and we denote the equivalence class containing p by $[p]$. We say that a connected digraph A is *simply connected* if any two paths in A with the same endpoints are homotopic.

(4) Let $(A;R)$ and $(B;R')$ be connected digraphs of the same type (that is, R, R' are either both symmetric or both anti-symmetric). A map $\sigma : A \longrightarrow B$ is a *homomorphism* if for every b, b' in the image of σ we have $R'(b,b')$ iff there exist $a, a' \in A$ with $R(a,a')$ and $\sigma a = b, \sigma a' = b'$. We say that a surjective homomorphism $\sigma : A \longrightarrow B$ is a *covering* of digraphs if for every $a \in A$ the restriction of σ to each of the sets a^+ and a^- is an isomorphism with each of the the subdigraphs on the sets $\sigma(a)^+$ and $\sigma(a)^-$ respectively.

(5) Now let x_0 be any element of L, and let $\mathcal{P}$ be the set of sequences in $\mathcal{S}$ which start at x_0. Let U be the set of homotopy classes of paths in $\mathcal{P}$. There is a well-defined map $\sigma : U \longrightarrow L$ given by $\sigma([x_0x_1 \dots x_r]) = x_r$. We now make U into a digraph (with edge relation S) and σ into a homomorphism of digraphs by setting $S([p_1],[p_2])$ if and only if $R(\sigma([p_1]),\sigma([p_2]))$ and $[p_2] = [p_1\sigma([p_2])]$. We shall abuse notation and also write $\sigma(x_0 \dots x_r) = x_r$. We refer to the homomorphism of digraphs $\sigma : U \longrightarrow L$ as the *universal covering* of L.

The last piece of terminology is justified by the following, which (for the anti-symmetric case) is Lemma 5.3 of [4].

Lemma 1.8 *Let L be a connected digraph.*

(i) Suppose $\tau : A \longrightarrow B$ is a covering of digraphs. If $p = x_0 \dots x_r$ is a path in B and $\tau(y_0) = x_0$ then there is a unique path p' in A starting at y_0 with $\tau(p') = p$. We call p' a lifting *of p. Moreover, if paths p_1, p_2 in B have the same starting points and are lifitngs of homotopic paths in A, then p_1, p_2 are homotopic paths in B.*

(ii) The universal covering $\sigma : U \longrightarrow L$ is a covering of digraphs and U is simply connected.

(iii) If $\tau : V \longrightarrow L$ is a covering, then there exists a surjective homomorphism $\Omega : U \longrightarrow V$ with $\sigma = \tau\Omega$. (In fact, Ω will be a universal covering.) Moreover, if V is simply connected, Ω will be injective. Thus any two simply connected coverings of L are isomorphic. $\square$

Lemma 1.9 *With the notation as in 1.8, any automorphism of L lifts to an automorphism of U.* $\square$

Definition 1.10 (1) With the notation as in 1.7, let $p = [x_0x_1 \dots x_rx_0] \in \sigma^{-1}x_0$. Define the *deck transformation* $d_p : U \longrightarrow U$ by

$$d_p([x_0x'_1 \dots x'_r]) = [x_0x_1 \dots x_rx_0x'_1 \dots x'_r].$$

Let $\Delta(L;R)$ be the set of all deck transformations.

(2) Let $\Gamma(L;R)$ be the group of automorphisms of the digraph U which preserve the set of fibres of σ.

Lemma 1.11 *With the notation as above, $\Delta(L;R)$ is the normal subgroup of $\Gamma(L;R)$ which consists of automorphisms of U fixing each fibre of σ.* $\square$

1.3 Graphic types

Definition 1.12 Suppose L is an irreducible transitive, permutation structure and that P is a 3-type and Q, R are 2-types such that

- $P(w,x,y)$ implies that $Q(w,x), Q(w,y)$ and $R(x,y)$;

- R is connected (i.e. the digraph given by R is connected);
- if $(w,x,y),(w',x,y) \in P$ then w,w' lie in the same H-orbit, whenever H is a closed subgroup of finite index in $Aut(L/x,y)$.

and *either*

- (*) if $R(x,y)$, $R(y,z)$ and $R(x,z)$ then there exists $w \in L$ such that $P(w,x,y), P(w,y,z)$ and $P(w,x,z)$;

or

- (**) $P = \{(w,x,y) \in L^3 : R(w,x), R(w,y), R(x,y)\}$.

Then we say that P is a *graphic* type (or that (P,Q,R) is a *graphic triple* of types) for L.

Theorem 1.13 *Let L be a transitive, irreducible permutation structure and suppose that (P,Q,R) is a graphic triple of types for L. Suppose $\tau : C \longrightarrow L$ is an untwisted, transitive, irreducible finite cover such that $Aut(C(x)/C(w)) = 1$ whenever $x,w \in L$ and $Q(w,x)$. Then there is an $Aut(C)$-invariant digraph relation R' on C such that $\tau : (C;R') \longrightarrow (L;R)$ is a covering of digraphs, and an automorphism g of the digraph $(C;R')$ is an automorphism of C if and only if g preserves the set of fibres of τ and g restricted to L is an automorphism of L.*

Proof. The relation R' is defined as follows.

> For $a,b \in C$, we have $R'(a,b)$ if and only if there exists $w \in L$ such that $P(w,\tau(a),\tau(b))$, and a,b lie in the same $Aut(C/C(w))$-orbit.

Note that this relation is invariant under $Aut(C)$.

Claim 1 If $P(w,\tau(a),\tau(b))$ and $P(w',\tau(a),\tau(b))$, and a,b lie in the same $Aut(C/C(w))$-orbit, then a,b lie in the same $Aut(C/C(w'))$-orbit.

Note that $\rho(Aut(C/C(\tau(a)),C(\tau(b))))$ is a closed subgroup of finite index in $Aut(W/\tau(a),\tau(b))$, so by the third property in the definition of a graphic type, there exists an automorphism $h \in Aut(C/C(\tau(a),\tau(b)))$ such that $hw = w'$. Let $g \in Aut(C/C(w))$ be such that $ga = b$. Then $h^{-1}gh \in Aut(C/C(w'))$ and $h^{-1}gha = b$.

Claim 2 $\tau : (C;R') \longrightarrow (L;R)$ is a digraph homomorphism.

Clearly, if $R'(a,b)$ then $R(\tau(a),\tau(b))$. Now suppose $x,y \in L$ and $R(x,y)$. As R is a type, there exists $w \in L$ such that $P(w,x,y)$. Let $a \in C(x)$. Because $Q(w,x)$ and $Q(w,y)$, and Q is a type, x and y lie in the same $Aut(L/w)$-orbit. As the cover is untwisted, it follows from Lemma 1.4 that they lie in the same $Aut(L/C(w))$-orbit. Thus there exists $b \in C(y)$ in the same $Aut(C/C(w))$-orbit as a, and so we have $R'(a, b)$.

Claim 3 For $c \in C$, the map $\tau \mid c^+$ is injective, with image $\tau(c)^+$.

Suppose $R'(c, a)$ and $R'(c, b)$ and $\tau(a) = \tau(b)$. By definition, there exist $w, w' \in L$ such that $P(w, \tau(c), \tau(a))$ and $P(w', \tau(c), \tau(b))$, and c, a lie in the same $Aut(C/C(w))$-orbit, and c, b lie in the same $Aut(C/C(w'))$-orbit. As $\tau(a) = \tau(b)$ it follows from Claim 1 that c, b lie in the same $Aut(C/C(w))$-orbit, and so a, b lie in the same $Aut(C/C(w))$-orbit. But $Aut(C(\tau(a))/C(w)) = 1$ and so $a = b$, as required. This gives injectivity. It is clear that $\tau(c^+) \subseteq \tau(c)^+$. The argument in Claim 2 gives the reverse inclusion.

Claim 4 For $c \in C$, the map $\tau \mid c^+$ is an isomorphism of digraphs.

First, note that if $a, b \in c^+$ and $R'(a, b)$ then $R(\tau(a), \tau(b))$. Conversely, suppose $a, b \in c^+$ and $R(\tau(a), \tau(b))$. We must show that $R'(a, b)$. The argument depends on which of (*) and (**) holds.

Suppose (*) holds. Then there exists $w \in L$ such that $P(w, \tau(c), \tau(a))$, $P(w, \tau(c), \tau(b))$ and $P(w, \tau(a), \tau(b))$. By Claim 1 (twice) a, b, c lie in the same $Aut(C/C(w))$-orbit. Thus $R'(a, b)$, by definition of R'.

Now suppose instead that (**) holds. Then $P(\tau(c), \tau(a), \tau(b))$, so (by Lemma 1.4) there is $b' \in C(\tau(b))$ which is in the same $Aut(C/C(\tau(c)))$-orbit as a. Then $R'(a, b')$. However, $R'(c, b')$ (because, $R'(c, a)$), so by Claim 3 $b = b'$, i.e. $R'(a, b)$.

Claim 5 For $c \in C$, the map $\tau \mid c^-$ is an isomorphism of digraphs with image $\tau(c)^-$.

We get injectivity, and the image of the map exactly as in Claim 3. It remains to show that if $a, b \in c^-$ and $R(\tau(a), \tau(b))$ then $R'(a, b)$. Well, by Claim 4, there exists $b' \in C(\tau(b))$ with $R'(a, b')$ and $R'(b', c)$. Thus, $\tau(b) = \tau(b')$ and $b, b' \in c^-$. So by injectivity, $b = b'$, whence $R'(a, b)$, as required.

Claim 6 The digraph $(C; R')$ is connected, so $\tau : (C; R') \longrightarrow (L; R)$ is a covering of digraphs.

Note that τ restricted to each connected component of $(C; R')$ is a covering of digraphs (as L is connected). So there are only finitely many connected components, and as C is transitive, these are permuted transitively by $Aut(C)$. The stabiliser of one of the components is a closed subgroup of finite index in $Aut(C)$ equal to the number of components, and so as $Aut(C)$ is irreducible, there is only one connected component, as required.

Claim 7 The automorphisms of C are determined by those of $(C; R')$ and of L.

Suppose $g \in Sym(C)$ preserves the fibres of τ, induces an automorphism of L, and preserves the digraph $(C; R')$. We have to show that $g \in Aut(C)$. As $\tau : C \longrightarrow L$ is a finite cover, we may assume that g fixes each fibre of τ, i.e. g

restricted to L is the identity. Now, as τ is untwisted, for $w \in L$ we have that $Aut(C(w)) = Aut(C(w)/L)$. In particular, $Aut(C(w)/L)$ is transitive on $C(w)$ and so we may assume that g fixes some element of $C(w)$. But now it follows from Claims 4, 5 and 6 that $g = 1$. □

Corollary 1.14 *Let L be an irreducible, transitive permutation structure and suppose that (P, Q, R) is a graphic triple of types for L. Let $\sigma : (U; R'') \longrightarrow (L; R)$ be the universal covering of digraphs, and let Δ be the group of deck transformations. Let Γ be the group of automorphisms of $(U; R'')$ which preserve the set of fibres of σ and induce automorphisms of L.*

Let $\tau : C \longrightarrow L$ be an untwisted, irreducible, transitive finite cover such that whenever $x, w \in L$ and $Q(w, x)$ we have $Aut(C(x)/C(w)) = 1$. Then there exists a surjective group homomorphism from Γ to $Aut(C)$ whose kernel H is contained in Δ, and such that Δ/H and $Aut(C/W)$ are isomorphic.

Conversely, if H is a normal subgroup of Γ which is of finite index in Δ and Γ/H is irreducible, then Δ/H is isomorphic to the kernel of some irreducible finite cover $\tau_H : C_H \longrightarrow L$.

Proof. Let $\tau : C \longrightarrow L$ be as in the statement, and let $(C; R')$ be the digraph given by Theorem 1.13. By Lemma 1.8(iii), there is a covering of digraphs $\kappa : (U; R'') \longrightarrow (C; R')$ such that $\sigma = \tau\kappa$. Note that as $(U; R'')$ is simply connected, κ is the universal covering of $(C; R')$, and so by Lemma 1.9 any automorphism of C lifts to an automorphism of $(U; R'')$. Let Γ_1 consist of the elements of Γ which preserve the set of fibres of κ. We claim that $\Gamma = \Gamma_1$. To prove this, as Γ_1 restricted to L is $Aut(L)$ (– any automorphism of L lifts to one of C), it is enough to prove that $\Delta \leq \Gamma_1$.

Let x_0 be the distinguished basepoint used in the construction of U (see 1.7(5)), and $c_0 = \kappa[x_0]$. Let p be a closed path in L starting and finishing at x_0; let $[x_0x_1 \dots x_n], [x_0y_1 \dots y_m] \in U$ be such that $\kappa([x_0x_1 \dots x_n]) = \kappa[x_0y_1 \dots y_m]$ (so $x_n = y_m$). We must show that $\kappa([px_0x_1 \dots x_n]) = \kappa[px_0y_1 \dots y_m]$. Let $c = \kappa([p])$. Now, $\kappa([px_0x_1 \dots x_n])$ is the endpoint of the unique lifting to C starting at c_0, of the path $px_0x_1 \dots x_n$. So this is the endpoint of the unique lifting starting at c of the path $x_0x_1 \dots x_n$. Likewise, $\kappa[px_0y_1 \dots y_m]$ is the endpoint of the unique lifting starting at c of the path $x_0y_1 \dots y_m$. By assumption, the liftings to C of the paths $x_0x_1 \dots x_n$ and $x_0y_1 \dots y_m$ which start at c_0 have the same endpoints. But $Aut(C/W)$ is transitive on $\tau^{-1}(x_0)$, so there is an element of $Aut(C/W)$ taking c_0 to c. Therefore the liftings to C starting at c of the paths $x_0x_1 \dots x_n$ and $x_0y_1 \dots y_m$ also have the same endpoints. This establishes the claim, and so gives the first part of the Corollary.

For the converse, suppose that H is as in the statement. Let C_H be the set of H-orbits on U. Define a digraph relation R' on C_H as follows. For distinct $a, b \in C_H$, set $R'(a, b)$ if and only if there exists $u \in a$ and $v \in b$ such that $R''(u, v)$. Define maps $\kappa_H : (U; R'') \longrightarrow (C_H; R')$ and $\tau_H : C_H \longrightarrow (L; R)$ by

$\kappa_H(u) = u/H$ and $\tau_H(u/H) = \sigma(u)$, where u/H denotes the H-orbit containing u. Each of these is a well-defined covering of digraphs. It is clear that the fibres of τ_H are finite, as Δ is transitive on each fibre of σ, and H is of finite index in Δ. Let $Aut(C_H)$ consist of permutations of C_H which preserve the relation R', the set of fibres of τ_H, and induce (via restriction) automorphisms of L (note that this is a closed subgroup of $Sym(C_H)$). We claim that τ_H is an irreducible finite cover of L.

First, note that any $g \in \Gamma$ induces an automorphism ϕg of C_H. The map $\phi g : C_H \longrightarrow C_H$ given by $\phi g(u/H) = (gu)/H$ is well-defined. Indeed, let $h \in H$ and $u' = hu$. Then $gu' = ghu = h'(gu)$ for some $h' \in H$ (as $H \trianglelefteq \Gamma$), so $gu/H = gu'/H$. Thus we have a homomorphism $\phi : \Gamma \longrightarrow Aut(C_H)$ (induced by κ_H) with kernel H. As κ_H is a covering ϕ is surjective (Lemma 1.9). Moreover, as σ is a covering, any automorphism of L lifts to an element of Γ, and so (using ϕ) to an element of $Aut(C_H)$. This shows that τ_H is a finite cover with automorphism group isomorphic to Γ/H, which is irreducible. Furthermore, it is clear that the kernel of τ_H is isomorphic to Δ/H. $\Box$

This result is not entirely satisfactory: for the converse, how are we to tell when Γ/H is irreducible? It would also be useful to know when τ_H is untwisted and locally trivial with respect to (w, x). We do not have a complete answer to these questions. However, in the next few lemmas we give conditions under which τ_H is always untwisted and locally trivial, and then analyse (with these hypotheses) what it means for Γ/H to be irreducible. For the rest of this subsection, the notation is as in 1.14.

Lemma 1.15 *(1) Suppose that $Aut(L/w)$ is irreducible (for $w \in L$). Then τ_H is untwisted.*

(2) Suppose that $Q = R$. Then τ_H is locally trivial with respect to (w, x).

(3) Suppose that $Aut(L/x, w)$ is irreducible. Then τ_H is locally trivial with respect to (w, x).

Proof. (1) This is immediate from Lemma 1.4.

(2) Suppose $R(w, x)$. As τ_H is a covering, for any $b \in C_H(w)$ there is a unique $a \in C_H(x)$ with $R'(b, a)$, and vice versa. It follows immediately that $Aut(C_H(x)/C_H(w)) = 1$.

(3) This is Lemma 1.5. $\Box$

Remark It appears that some extra hypothesis is necessary here, as the universal covering $\sigma : (U; R'') \longrightarrow L$ is not necessarily locally trivial. For example, this happens if there is a path p from w to x and $g \in Aut(L/w, x)$ such that p and gp are not homotopic.

Now we turn to the question of when Γ/H is irreducible. Clearly this is the case if and only if for every closed subgroup Σ of finite index in Γ we have $\Sigma H = \Gamma$ (alternatively, $(\Sigma \cap \Delta)H = \Gamma$). The next lemma gives a description of the proper, closed subgroups of finite index in Γ. In particular, if $(L;R)$ is not triangle-free, there are no such subgroups, and the converse statement in Corollary 1.14 holds without the assumption that Γ/H be irreducible.

Lemma 1.16 *With the notation as in Corollary 1.14, suppose additionally that $Aut(L/w)$ is irreducible, and that if H is a normal subgroup of Γ contained in Δ then τ_H is locally trivial with respect to (w,x), whenever $Q(w,x)$ holds (for example, as a result of Lemma 1.15). Let Σ be a closed normal subgroup of finite index in Γ.*

(1) If $(L;R)$ is not triangle-free, then $\Sigma=\Gamma$.

(2) If $(L;R)$ is triangle-free, then $(U;R'')$ is a discrete tree, and if $\Sigma\neq\Gamma$ then it is the stabiliser of a class of one of the finite equivalence relations E_n on $(U;R)$ described below:

(i) If $(L;R)$ is triangle-free and R is symmetric, then $n=2$ and for a,b in the tree $(U;R'')$, we have $E_2(a,b)$ iff the length of the shortest path in $(U;R'')$ from a to b is even.

(ii) Suppose $(L;R)$ is triangle-free and anti-symmetric. Let $n\in\mathbf{N}$. Let $a,b\in U$ and let $x_1x_2\ldots x_m$ be a path in $(U;R)$ from $x_1=a$ to $x_m=b$. For $1\leq i\leq m-1$ let $\epsilon_i=1$ if $R(x_i,x_{i+1})$ and $\epsilon_i=-1$ otherwise. Let

$$\eta(a,b)=\sum_{i=1}^{m-1}\epsilon_i.$$

This is independent of the path chosen, and we set $E_n(a,b)$ iff $\eta(a,b)$ is divisible by n.

Proof. Let $H=\Sigma\cap\Delta$, and let τ_H be as constructed in the proof of Corollary 1.14. Then τ_H splits. Consider the Σ-orbits on C_H. Arguing as in Lemma 3.2 of [4], each such orbit contains one element of each fibre of τ_H, and the orbits form an $Aut(C_H)$-invariant partition of C_H. We consider the two cases separately.

(1) We claim that some edge of C_H has both its endpoints in the same Σ-orbit. By hypothesis, $(L;R)$ is not triangle-free, so (as τ_H is a covering) there exist $a,b,c\in C_H$ such that $R'(a,b),R'(a,c)$ and $R'(b,c)$. As $(L;R)$ is edge-transitive, and τ_H is a covering, there is an element $g\in Aut(C_H/a)$ with $gb=c$. But because $Aut(L/\tau_H(a))$ is irreducible, and both Σ and $Aut(C_H/a)$ intersect the kernel of τ_H trivially, we have $g\in\Sigma$, which gives the claim. Thus there are no edges between different Σ-orbits (by edge-transitivity of $Aut(C_H)$). So either Σ is transitive, in which case the cover is trivial, or C_H is not connected. The latter is impossible, as U is connected, and C_H is a homomorphic image of U.

(2) By the remarks at the start of the proof, if n is the index of Σ in Γ, then there are n Σ-orbits on U, and these are permuted regularly by Γ. Let E be the finite equivalence relation with classes the Σ-orbits. We have to show that $E = E_n$. The argument obviously depends on whether R is symmetric or not.

(i) Suppose R is symmetric. Let abc be a path of length 2 in U. Then by edge-transistivity of $(U; R'')$, there exists an automorphism of U fixing b and sending a to c. This obviously fixes the E-eqivalence class containing b, and so by regularity of Γ on the E-classes, fixes the E-class of a. Thus $E(a, c)$. So alternate vertices along any path are in the same E-class, and (i) now follows.

(ii) Suppose R is anti-symmetric. We leave the reader to check that E_n as defined above is indeed a Γ-invariant equivalence relation on U with n classes. Thus it will suffice to show that E refines E_n. By edge-transitivity, and the argument in (i), if $R''(a, b)$ and $R''(a', b)$ then $E(a, a')$. Similarly, if $R''(b, c)$ and $R''(b, c')$ then $E(c, c')$. It follows that if $\eta(a, b) = \eta(a, b')$ then $E(b, b')$. So as $\eta(a, b) = \eta(a, c) + \eta(c, a)$ for any $a, b, c \in U$, it will suffice to show that if $\eta(a, b) = n$ then $E(a, b)$.

Label the classes of E as $E^1 \dots E^n$. Write $f(i, j)$ if there exists $a \in E^i$ and $b \in E^j$ with $R''(a, b)$. Then f is a permutation of $\{1, \dots, n\}$. In fact, f is an n-cycle: if $f^k(i) = i$ for some $1 \leq k < n$ and $i \in \{1, \dots, n\}$ then $E(a, b)$ for a, b with $\eta(a, b) = k$, so E has $< n$ classes.

Let $a_0 x_1 \dots a_n$ be a path in U with $R''(a_i, a_{i+1})$ for $0 \leq i \leq n-1$. Then $E(a_0, a_n)$, by the previous paragraph. This finishes the proof of the lemma. □

1.4 The homology module

A basic result of algebraic topology is that the first homology group of a path-connected space is the abelianisation of its fundamental group. We adapt this result in a straightforward way to construct, as a section of some permutation module, an abelian group $H(L)$ which has as its finite homomorphic images the kernels of irreducible, locally trivial finite covers of L, where L is a countable, permutation structure satisfying the hypotheses of Theorem 1.13 . As an easy application of this, we show (Corollary 1.18) that if L is $\aleph_0$-categorical, then the rank of these kernels is bounded. This gives a partial answer to the question at the end of [4]. We will obtain a more complete answer in Section 3 by investigating the difference between locally trivial, and superlinked covers.

Let L be a transitive, permutation structure with a graphic triple of types (P, Q, R). Let $\overline{R} = \{(a, b) \in L^2 : R(a, b) \text{ or } R(b, a)\}$. Let F be the free abelian group with basis $\overline{R}$ and F_1 the subgroup generated by $\{(a, b) + (b, a) : R(a, b)\}$. Let $A = F/F_1$. So $(a, b) = -(b, a)$ in A. (Note that A is free abelian, with basis 'half' of $\overline{R}$.) Define a function $\phi : \mathcal{S} \longrightarrow A$ by

$$\phi(x_0 x_1 \ldots x_k) = \sum_{i=0}^{k-1} (x_i, x_{i+1}),$$

where $\mathcal{S}$ is the set of closed paths in $(L; R)$. Let D be the subgroup of A generated by $\phi(\mathcal{S})$ and E the subgroup

$$\langle (x_0, x_1) + (x_1, x_2) + (x_2, x_0) : R(x_0, x_1), R(x_1, x_2), R(x_0, x_2) \rangle.$$

Note that there is a natural action of $Aut(L)$ on A, and D, E are invariant under this. So we get an action of $Aut(L)$ on D/E. We define $H(L)$ to be the largest quotient of this on which $Aut(L)$ acts trivially, that is $H(L) = D/E^1$ where $E^1 = E + \langle gd - d : g \in Aut(L), d \in D \rangle$.

If $(L; R)$ is triangle-free, then $E = \{0\}$ and there are some natural $Aut(L)$-invariant subgroups of finite index in A. Suppose first that R is symmetric. There is a homomorphism of abelian groups $\alpha : F \longrightarrow \mathbf{Z}/2\mathbf{Z}$ given by

$$\alpha(\sum_{e \in R} a_e e) = \sum_{e \in R} a_e \pmod 2.$$

F_1 is in the kernel of α, and so α induces a homomorphism for A to $\mathbf{Z}/2\mathbf{Z}$, which is clearly invariant under that action of $Aut(L)$ on A. Let A^0 be the kernel of this, and $D^0 = A^0 \cap D$. Note that this contains E^1 (it is of finite index in D and is $Aut(L)$-invariant).

Now suppose that R is anti-symmetric. Then write $\overline{R}$ as the disjoint union of R and R^-. Define a homomorphism of abelian groups $\alpha : F \longrightarrow \mathbf{Z}$ by

$$\alpha(\sum_{r \in R} a_r r + \sum_{r' \in R^-} b_{r'} r') = \sum_{r \in R} a_r - \sum_{r' \in R^-} b_{r'}.$$

Again, F_1 is in the kernel of α so α induces a homomorphism from A to $\mathbf{Z}$, which is invariant under the action of $Aut(L)$ on A. For $n \in \mathbf{N}$ let α_n be the result of composing α with reduction of $\mathbf{Z}$ modulo n. Let A^n be the kernel of α_n and let $D^n = A^n \cap D$. Again, this contains E^1.

Theorem 1.17 *Suppose $\tau : C \longrightarrow L$ is a Q-locally trivial, irreducible finite cover with kernel K. Then K is a homomorphic image of $H(L)$.*

Conversely, suppose that the additional hypotheses of Lemma 1.16 hold. Let H be a subgroup of $H(L)$ such that one of the following holds.

- *$(L; R)$ is not triangle-free;*
- *$(L; R)$ is triangle-free, and R is symmetric, and $H + D^0/E^1 = H(L)$;*
- *$(L; R)$ is triangle-free, R is anti-symmetric, and for every $n \geq 2$ we have $H + D^n/E^1 = H(L)$.*

Then there exists a Q-locally trivial, irreducible finite cover of L with kernel isomorphic to D/H.

Proof. Once we have Corollary 1.14, the proof is really just a copy of the standard result of algebraic topology mentioned above (see, for example [7] p.50). However, as there is no obvious reference for the precise statement we need, we include the argument here.

Recall that U is the set of homotopy classes of paths in $(L;R)$ which start at x_0. Let U_0 be the subset consisting of those which also finish at x_0.

Claim 1 If p, p' are homotopic closed paths in $(L;R)$ then $\phi(p)+E = \phi(p')+E$.

It is enough to check this for elementary homotopies, and we leave this to the reader.

So we have a well-defined map $\psi : U_0 \longrightarrow D/E$ given by

$$\psi([p]) = \phi(p) + E$$

for $[p] \in U$. There is a natural bijection between Δ, the group of deck trasformations of the universal covering $\sigma : (U;R'') \longrightarrow (L;R)$ and U_0. With this identification, we get a map $\psi : \Delta \longrightarrow D/E$, and it is easy to check that this is a surjective group homomorphism.

Claim 2 The kernel of ψ is equal to the derived subgroup of Δ.

As the image of ψ is abelian, it is clear that its kernel contains the derived group $\Delta^{(1)}$ of Δ. So suppose $[p] \in ker\psi$. We must show that $[p] \in \Delta^{(1)}$.

Write $p = x_0 x_1 \dots x_k x_0$. Then

$$\phi(p) = \sum_{j=1}^{m} \alpha_j((y_{j0}, y_{j1}) + (y_{j1}, y_{j2}) - (y_{j0}, y_{j2}))$$

for some $\alpha_j \in \mathbf{Z}$ and points y_{ji} with $R(y_{j0}, y_{j1}), R(y_{j1}, y_{j2}), (y_{j0}, y_{j2})$. Let $x_0 z_{ji} y_{ji}$ be a path from x_0 to y_{ji} (and choose these so that if $y_{ji} = y_{j'i'}$ then $z_{ji} = z_{j'i'}$). Consider

$$p_{j0} = x_0 z_{j0} y_{j0} y_{j1} \overline{z_{j1}} x_0$$

$$p_{j1} = x_0 z_{j1} y_{j1} y_{j2} \overline{z_{j2}} x_0$$

$$\overline{p_{j2}} = x_0 z_{j0} y_{j0} y_{j2} \overline{z_{j2}} x_0.$$

(Here, the overline denotes the reverse path.) Then let $p_j = p_{j0} p_{j1} p_{j2}$. Clearly $[p_j] = [x_0]$, the identity in Δ. Let $\Omega : \Delta \longrightarrow \Delta/\Delta^{(1)}$ be the natural map. Without loss of generality, the x_i's are amongst the $y_{ji'}$'s. So

$$[p] = \prod_{i=0}^{k-1} [x_0 z_i x_{i+1} \overline{z_{i+1}}\; x_0]$$

where if $x_i = y_{ji'}$then $z_i = z_{ji'}$. Write $q_i = x_0 z_i x_{i+}\overline{z_{i+1}}\, x_0$. So

$$\Omega[p] = \prod_{i=0}^{k-1} \Omega[q_i].$$

Now, we can rearrange the order of the $\Omega[q_i]$'s and (because of the equation for $\phi[p]$)

$$\Omega[p] = \prod \Omega[q_i] = \prod (\Omega[p_j])^{\alpha_j}$$

which is the identity. So $[p] \in \Delta^{(1)}$ as required.

So $\Delta/\Delta^{(1)}$and D/E are isomorphic abelian groups. They are both $Aut(L)$-modules (the action in the first is that given by conjugation in $\Gamma/\Delta^{(1)}$), and the isomorphism is actually an isomorphism of $Aut(L)$-modules. The result now follows from Corollary 1.14, and the lemmas which follow it. □

The following is a straightforward corollary to Theorem 1.17, and provides a partial answer to the question at the end of [4].

Corollary 1.18 *Let L be a transitive, countable $\aleph_0$-categorical structure, and suppose that (P, Q, R) is a graphic triple of types for L. Then:*

(1) $H(L)$ is finitely generated;

(2) there exists an integer r such that the rank of the kernel of any Q-locally trivial, irreducible finite cover of L is at most r.

Proof. By Theorem 1.17, it is enough to prove the first of these. We use the notation introduced at the start of this subsection.

As $(L; R)$ is connected and $\aleph_0$-categorical, it follows from the Ryll-Nardzewski theorem that $(L; R)$ has finite diameter, d.

Claim 1 D is generated by

$$\{\phi(p) : p \,\text{is a closed path of length at most}\, 2d + 1\}.$$

Let $p = x_0 x_1 \ldots x_t$ be a closed path in $(L; R)$ (so $x_t = x_0$) and suppose that $t > 2d + 1$. There exists a path $x_0 x'_1 \ldots x'_{k-1} x_{d+1}$, where $k \leq d$. Let

$$p' = x_0 x_1 \ldots x_d x_{d+1} x'_{k-1} \ldots x'_1 x_0$$

and

$$p'' = x_0 x'_1 \ldots x'_{k-1} x_{d+1} x_{d+2} \ldots x_t.$$

Then p' has length $d + 1 + k$ and p'' has length $k + (t - (d + 1))$. Thus, p' has length at most $2d + 1$, and p'' has length less than t. As $\phi(p) = \phi(p') + \phi(p'')$, the claim follows by induction.

Claim 2 Let $D' = \langle gz - z : g \in Aut(L), z \in D \rangle$. Then D/D' is a finitely generated abelian group.

Let $p_1, \ldots p_r$ be representatives from the orbits of $Aut(L)$ on closed paths of length at most $2d + 1$ in $(L; R)$. Note that r is finite here, because of $\aleph_0$-categoricity of L. Then

$$D/D' = \langle \phi(p_i) + D' : i = 1, \ldots, r \rangle.$$

Indeed, let p be any closed path of length at most $2d+1$. Then there exists $i \leq r$ and $g \in Aut(L)$ such that $p = gp_i$. Then $\phi(p) = g\phi(p_i)$. But $\phi(p_i) - g\phi(p_i) \in D'$. So $\phi(p) + D' = \phi(p_i) + D'$, and the statement, and hence the claim, now follow from Claim 1.

As $H(L)$ is a homomorphic image of D/D', we now have (1). □

2 Finding graphic triples and computing $H(L)$

2.1 Strong types

Definition 2.1 Suppose W is a permutation structure. A *strong type over* W is a non-constant function p which to each finite $A \subseteq W$ assigns a 1-type over A, denoted by $p \mid A$, such that

- if $A \subseteq A'$ then $p \mid A' \subseteq p \mid A$;
- if $g \in Aut(W)$ then $g(p \mid A) = p \mid gA$.

Remark Assuming here that p is non-constant is equivalent to insisting that $(p \mid A) \cap A = \emptyset$ for all A.

Lemma 2.2 *Let p be a strong type over W and A a finite subset of W. Let H be a closed normal subgroup of finite index in $Aut(W/A)$. Then H is transitive on $p|A$.*

Proof. The H-orbits on $p|A$ form an $Aut(W/A)$-invariant partition of $p|A$ with finitely many parts, and these parts are permuted transitively by $Aut(W/A)$. Now, H is also an open subgroup of $Aut(W/A)$, so there exists a finite subset X of W such that $Aut(W/A \cup X) \leq H$. Thus $p \mid A \cup X$ is contained in one of the H-orbits, O, say. Suppose now that H is not transitive on $p|A$. Then there exists $g \in Aut(W/A)$ such that $gO \cap O = \emptyset$. Thus

$$(p|(A \cup X)) \cap (p|(A \cup gX)) = \emptyset$$

which is impossible. So H is transitive on $p|A$, as required. □

Lemma 2.3 *Let W be a transitive, irreducible permutation structure. Suppose that p is a strong type over W. Let*

$$R = \{(x, y) \in W^2 : x \in p \mid \{y\}\}$$

and

$$P = \{(w, x, y) \in L^3 : R(x, y) \text{ and } w \in p \mid \{x, y\}\}.$$

Then

(1) (P, R, R) is a graphic triple of types for W.

(2) Any two paths in $(W; R)$ which have the same endpoints are homotopic.

Proof. (1) As W is transitive, both R and P are types. Suppose $P(w, x, y)$. Then $w \in p \mid \{x, y\} \subseteq p \mid \{y\}$ so $R(w, y)$. It is clear that $(W; R)$ is connected, with diameter 2. Thus it remains to show that if $R(x, y)$ then $\{w : P(w, x, y)\}$ is an H-orbit, for any closed subgroup H of finite index in $Aut(W/x, y)$, and this follows from the previous lemma.

(2) Let $x_0 x_1 \ldots x_k$ be a path in $(W; R)$. Let $w \in p \mid \{x_0, x_1, \ldots, x_k\}$. So $R(w, x_i)$ for $0 \leq i \leq k$. It will suffice to show that the original path is homotpic to $x_0 w x_k$. This reduces to the case where $k = 1$, and this is straightforward: for example, if $R(x_0, x_1)$ we have elementary homotopies

$$x_0 x_1 \rightsquigarrow x_0 w x_0 x_1 \rightsquigarrow x_0 w x_1. \square$$

We shall call (P, R, R) the *graphic triple of the strong type p*.

Corollary 2.4 *Let W be a transitive, irreducible permutation structure. Suppose W has a strong type P with graphic triple (P, R, R). Then any untwisted, R-locally trivial irreducible finite cover of W is trivial.*

Proof. This follows immediately from Lemma 2.3 and Theorem 1.13. $\square$

Suppose that W is a stable first-order structure, and assume that W is also saturated, transitive and irreducible. Then the finite equivalence relation theorem of Shelah gives that over every (finite) subset A of W there is a unique non-forking extension of p_0, the 1-type over $\emptyset$. If we let $p \mid A$ be the set of realistations in W of this non-forking extension, then p is clearly a strong type in the sense of Definition 2.1 (of course, the definition 2.1 *originated* in stability theory!). Thus Corollary 2.4 is applicable in this case. However, Theorem 1.13 is not really needed here; we can use the finite equivalence relation theorem directly:

Lemma 2.5 *Let M, W be transitive, irreducible, stable saturated structures. Let $\tau : M \longrightarrow W$ be an irreducible, transitive finite cover such that if $w, x \in W$ are independent then $Aut(M(x)/M(w)) = 1$. Then τ is one-to-one.*

Proof. Suppose τ is not one-to-one. Let $x, w \in W$ be independent. Choose $a, b \in M(x)$ with $a \neq b$. By transitivity of M, $tp(a/\emptyset) = tp(b/\emptyset)$. Now, both a and b are independent from $M(w)$ in M. But by assumption (and saturation), $tp(a/M(w)) \neq tp(b/M(w))$. As these are non-forking extensions of the same type over $\emptyset$, there exists a finite equivalence relation E definable over $\emptyset$ such that

$$M \models \neg E(a, b).$$

Then the stabiliser in $Aut(M)$ of the E-classes is a proper closed subgroup of finite index in $Aut(M)$, which contradicts irreducibility of M. $\square$

Remark In the original version of this paper, the conclusion in the above lemma was that $Aut(M/W) = 1$. The additional refinement was pointed out to us by J. Koshan and A. Lachlan.

2.2 Examples

Example 2.6 The following problem arises when one tries to make more precise the main results (3.1 and 4.7) of [5]. Let D be a countable $\aleph_0$-categorical structure and X a finite algebraically closed subset of D. Let W be a countable structure biinterpretable (in the model-theoretic sense) with $(D, x)_{x \in X}$ (i.e. D with the elements of X named). So W is $\aleph_0$-categorical and $Aut(W)$ is topologically isomorphic to $Aut(D/X)$. Let Γ be a closed permutation group on a set C and suppose $\rho : \Gamma \longrightarrow Aut(W)$ is a non-split, continuous epimorphism with finite kernel. Describe the possibilities for the kernel of ρ.

We shall show that for various D, the kernel of ρ as above must always be trivial. We will assume that D is transitive, and has weak elimination of imaginaries (cf. 1.3 of [5]). So $Aut(W)$ is irreducible, and we may assume that Γ is also irreducible. Furthermore, we may assume that Γ is transitive on C and $Aut(W)$ is transitive on W (any $\aleph_0$-categorical structure is biinterpretable with a transitive one). The $ker\rho$-orbits on C are finite and form a Γ-invariant partition of C, and the permutation group induced on these orbits is topologically isomorphic to $Aut(W)$. Thus we can assume that these orbits are the fibres of an irreducible, superlinked finite cover $\pi : C \longrightarrow W$, and that ρ is the associated restriction map.

We wish to replace π by a cover $\pi' : C' \longrightarrow W'$ biinterpretable with it, in which the stabiliser of a point in W' is irreducible. So suppose that for some non-empty, finite algebraically closed subset Z of W, $Aut(W/Z)$ is irreducible. Let z be an enumeration of Z and y an enumeration of $\pi^{-1}(Z)$. Let W', C' and π' be respectively the Γ-orbits containing z, y and (y, z). Then Γ is a closed permutation group on C', and $\pi' : C' \longrightarrow W'$ is a finite cover biinterpretable with π.

Because D has weak elimination of imaginaries, the permutation structure $\langle W'; Aut(W)\rangle$ is isomorphic (as a permutation group) to $\langle W''; Aut(D/X)\rangle$, where W'' is an $Aut(D/X)$-orbit containing an enumeration of some algebraically closed subset $Y \supseteq X$. We now find strong types for these structures. Here are some instances where the steps in the above programme can be caried out without too much effort.

Theorem 2.7 *Suppose D is one of the following countable $\aleph_0$-categorical structures.*

(i) a pure set;

(ii) the rational numbers as an ordered set;

(iii) the universal, homogeneous, countable poset;

(iv) the universal, homogeneous, countable graph;

(v) the universal, homogeneous, countable, K_n-free graph;

(vi) the universal, homogeneous, countable tournament;

(vii) the universal, homogeneous, countable digraph;

(viii) a Henson digraph (that is, a homogeneous countable, digraph with age the collection of all finite digraphs which do not embed any of a set of finite tournaments);

(ix) a vector space over a finite field.

Let X be a finite algebraically closed subset of D. Suppose Γ is a closed permutation group on a countable set, and $\rho : \Gamma \longrightarrow Aut(D/X)$ a continuous, non-split, epimorphism with finite kernel. Then ρ is an isomorphism.

Proof. Each of these structures has weak elimination of imaginaries. Moreover, if D is one of them, X a finite subset of D, and $z \in D^{eq}$ there exist $z_1, \ldots, z_m$, $Aut(D/X)$-translates of z, such that $Aut(D/X, z_1, \ldots, z_m)$ is irreducible. Indeed, by weak elimination of imaginaries, it is enough to prove this when z is a finite algebraically closed subset of D containing X. In cases (i) – (viii), algebraic closure is trivial, and so for every $x \in z \setminus X$ there exists $g \in Aut(D/X)$ with $z \cap gz = z \setminus \{x\}$. We leave it as an exercise for the reader to deal with the cases in (ix). So by the discussion above and Corollary 2.4 and Lemma 1.5, it will be enough to exhibit a strong type p for W'' where W'' consists of an $Aut(D/X)$-orbit containing an enumeration of some algebraically closed subset $Y \supseteq X$, and show that, if (P, R, R) is the associated graphic triple of types and $w, x \in W''$ are such that $R(w, x)$, then $Aut(W''/w, x)$ is irreducible. This latter point follows automatically from the weak elimination of imaginaries, and the fact that algebraic closure is equal to definable closure in all of these structures. So all that is left to do is to exhibit the strong types in each case.

(i) If A is a finite subset of W'' then $p|A$ consists of those elements $z \in W''$ whose intersection with each element of A is (an enumeration of) X.

(ii) The type over X of a tuple of elements of $\mathbf{Q}$ is determined by how the elements lie in the intervals of $\mathbf{Q} \setminus X$. For $x, y \in W''$ say that $x < y$ iff the elements of x not in X come before those of y in each of these intervals. Then, for A a finite subset of W'' let $p|A$ be those elements of W'' which are $<$ every element of A.

(iii) Exercise.

(iv), (v), (vii), (viii) If A is a finite subset of W'', let $p|A$ consist of those elements x of W'' such that if $y \in A$ then there are no edges between elements of x not in X and elements of y not in X.

(vii) If A is a finite subset of W'', let $p|A$ consist of those elements x of W'' such that if $y \in A$ then there any element of x not in X dominates any element of y not in X.

(ix) For the vector space case, let $p|A$ be the set of elements of W'' which are independent from A over X. $\square$

Example 2.8 In Theorem 5.8 of [4] we constructed (as digraph coverings) non-split superlinked finite covers of the universal, countable, homogeneous local order L. It follows from Theorem 1.13, Lemma 1.5, and the description of the universal covering of L given in [4], that these are all the irreducible, superlinked finite covers of L.

Example 2.9 In [13], Ivanov investigates minimal finite covers of the countable, homogeneous, universal circular order W. He shows that for each natural number n there is a unique transitive, irreducible superlinked finite cover of W with kernel the cyclic group of order n. The construction of these is similar to the construction of the superlinked covers of the local order in [4]. Our Theorem 1.13 is not directly applicable here as there is no graphic triple of types on W. However, we can apply our result indirectly (if somewhat inelegantly), as we now indicate brieΩy.

Let W' be the permutation structure with domain the set of ordered pairs of distinct elements from W, and automorphisms those permutations induced by elements of W. Let R be the 2-type on W' given by

$$R((a,b),(c,d)) \text{ iff } \zeta(a,b,c) \wedge \zeta(b,c,d) \wedge \zeta(c,d,a),$$

where ζ is the circular ordering on W. This gives a graphic triple of types on W', using (**) in Definition 1.12. It can be shown that the homology group $H(W')$ is isomorphic to $\mathbf{Z}$ (in fact, one can easily exhibit the universal covering of $(W';R)$, as we did for the local order in [4]). As $Aut(W'/w,x)$ is irreducible for any $w, x \in W'$, this shows that kernels of irreducible superlinked finite covers of W' are precisely the finite cyclic groups. Any irreducible superlinked finite cover of W gives rise to one of W', and so to finish our derivation of Ivanov's result, we must show that if $\pi : C \longrightarrow W'$ is a superlinked irreducible finite cover of W' then it arises from one of W. Let $\Sigma = Aut(W')$ and $\Gamma = Aut(C)$. There is a topological isomorphism $\phi : \Sigma \longrightarrow Aut(W)$. Let $a \in W$ and $\Sigma_1 = \phi^{-1}(Aut(W/a))$. Note that Σ_1 is topologically isomorphic to $Aut(\mathbf{Q},<)$. By Theorem 2.7, $\rho^{-1}(\Sigma_1)$ splits over $ker\rho$. So there is a closed subgroup Γ_1 of Γ which is disjoint from $ker\rho$ and $\phi\rho(\Gamma_1)$ is equal to $Aut(W/a)$. Then, by considering the permutation structure with domain the cosets of Γ_1 in Γ and automorphisms given by the action of Γ on this, we get a finite cover of W which is biinterpretable with π (so is irreducible and has kernel the same as π).

Example 2.10 Let W be a countable 2-homogeneous tree which is not a semilattice. In Theorem 4.3 of [4] we showed that if W has infinite ramification index, then any finite cover of W splits. We now indicate how to remove this restriction on the ramification index. First note that both $Aut(W)$ and the stabiliser of a point in W are irreducible. Furthermore, algebraic closure in W is trivial. So by Lemma 2.5 of [4], to show that every finite cover of W splits, it is enough to show that a transitive, irreducible superlinked finite cover $\pi : C \longrightarrow W$ with simple fibre group is trivial. Now, for each finite subset A of W, let

$$P \mid A = \{x \in W : x < a\, \forall a \in A\}.$$

This is a strong type. Moreover, if $w < x$ then $Aut(W/w, x)$ is irreducible, and so π is locally trivial with respect to (w, x). Thus by Corollary 2.4, π is trivial.

Example 2.11 Recall that a *C-relation* is obtained from a tree $(T, <)$ by taking a set W of maximal chains which cover the tree, and a 3-ary relation $\Omega(-;-,-)$ on this set given by

$$\Omega(a; b, c) \Leftrightarrow a \cap b \subset b \cap c.$$

If $(T, <)$ is 2-homogeneous, the structure (W, Ω) is homogeneous (in the sense of Fraïssé). In particular, it is 2-transitive. We claim that any finite cover of W splits. As in the previous example, algebraic closure is trivial, and $Aut(W)$ and the stabiliser of a point of W are both irreducible, and so it will suffice to show that any irreducible, transitive, superlinked finite cover $\pi : C \longrightarrow W$ with simple fibre group is trivial.

As the stabiliser of a point is irreducible, π is untwisted. The fibre group is simple, and so, for distinct $w, x \in W$, π is either locally trivial, or locally transitive with resprct to (w, x). Results in the next section will enable us to rule out the latter possibility, so assume for the moment that we know π to be locally trivial. Then by Corollary 2.4, π is trivial, because

$$p|A = \{a : \Omega(a; b, c)\, \forall b, c \in A\}$$

is a strong type for W (where A is a finite subset of W).

To complete this example, we now assume that the original π is locally transitive and non-trivial, and deduce a contradiction. To do this, we look ahead to Lemma 3.3. Let K be the kernel of π. Then for distinct $w, x \in W$ there is a continuous epimorphism $\phi_{w,x} : Aut(W/w, x) \longrightarrow K$ (and conjugation in $Aut(W)$ permutes these maps). It follows that K is the cyclic group of order 2; the ramification index of the original tree $(T, <)$ is finite; and for $g \in Aut(W/w, x)$ the image $\phi_{w,x}(g)$ is determined by the parity of the permutation induced by g on the cones above $w \cap x$ in T. Now, suppose $w, x, y \in W$ and $\Omega(y; w, x)$. Let

$g \in Aut(W/w, x, y)$. Then from what we have just observed, $\phi_{x,y}(g) = \phi_{w,x}(g)$. It now follows from Lemma 3.3(2) that $\phi_{w,x}(g) = 1$. However, any permutation of the cones above $w \cap x$ can be effected whilst fixing y, and so this implies that $K = 1$, a contradiction.

Example 2.12 We prove the following result on perfect central extensions of the symmetric group which was mentioned in the Introduction. Clearly a similar result holds for some of the other automorphism groups mentioned in Theorem 2.7, provided they have the small index property.

Theorem 2.13 *Suppose Γ is a perfect group having a finite, central subgroup Z such that Γ/Z is isomorphic to the symmetric group on a countably infinite set. If Γ has a subgroup U of index $< 2^{\aleph_0}$ which is disjoint from Z, then $Z = 1$.*

Proof. Let D be a countably infinite set, and $\alpha : \Gamma \longrightarrow Sym(D)$ an epimorphism with kernel Z. Note that $\alpha(U)$ is of index $< 2^{\aleph_0}$ in $Sym(D)$, and so by the small index property, it contains the stabiliser of some n-tuple of elements of D, for some n. By taking a subgroup of U if necessary, we can assume there is equality (rather than just containment) here. Let C be the set of cosets of U in Γ. Γ acts faithfully on this set, and so we can consider Γ as a subgroup of $Sym(C)$. We consider the permutation structure on C with automorphism group the closure of Γ in C. Let W be the set of Z-orbits on C and $\pi : C \longrightarrow W$ be given by $\pi(a) = Za$. Now, the permutation structure given by the action of Γ on W is isomorphic to $Sym(D)$ acting on an orbit of n-tuples. This is closed, so π is a finite cover.

We now show that Γ is actually equal to $Aut(C)$ (at present, we only know that it is dense in it). It is enough to show that $Aut(C/W) = Z$. Let x, w be distinct elements of W. Then $\Gamma_{C(w)} \cap Z = 1$ and so α restricted to $\Gamma_{C(w)}$ is injective. Both $\alpha\Gamma_{x,C(w)}$ and $\alpha\Gamma_{C(x),C(w)}$ are of finite index in $Aut(W/w, x)$ and so equal it. Thus $\Gamma_{x,C(w)} = \Gamma_{C(x),C(w)}$, and so (as Γ is dense in $Aut(C)$) $Aut(C(x)/C(w)) = 1$. It follows that $Aut(C/W, C(w)) = 1$ and so $Aut(C/W) = Z$, as required. So π is superlinked finite cover of W with kernel Z and automorphism group isomorphic to Γ. As Γ is perfect it has no proper subgroups of finite index, so π is irreducible. The theorem now follows from 2.7. $\square$

3 Superlinked covers which are not locally trivial

By Theorem 1.13, if W is a transitive, irreducible permutation structure with a graphic triple of types (P, Q, R), then any irreducible, Q-locally trivial finite cover of W can be described as a covering of the digraph $(W; R)$. Furthermore, by Lemma 1.5 we know that if $Aut(W/w, x)$ is irreducible for $(w, x) \in Q$, then any superlinked irreducible finite cover is Q-locally trivial. So in this case we have a complete picture of the superlinked finite covers. However, it seems rather unsatisfactory to simply impose the extra irreducibility hypothesis and consider that we have the end of the story: the most natural example of an irreducible superlinked finite cover is a vector space covering its projective space, and this is not locally trivial (for any graphic type). So there must be some other mechanism for producing irreducible, superlinked finite covers, and it is this which we shall investigate in this section.

3.1 A factorisation

The following lemma, which can be regarded as a more precise version of Lemma 1.5, shows that a superlinked finite cover of W can be factored as a locally trivial cover, followed by a finite cover whose kernel is a homomorphic image of the stabiliser of two points of W.

Lemma 3.1 *Let W be a transitive, irreducible permutation structure. Let $\pi : C \longrightarrow W$ be an untwisted, transitive, irreducible, superlinked finite cover, and $x, w \in W$. Then there exist finite covers $\pi_1 : C \longrightarrow C'$ and $\pi_2 : C' \longrightarrow W$ such that*

- $\pi = \pi_2\pi_1$;
- $Aut(C'(x)/C'(w)) = 1$;
- *if $s \in C'(x)$ and $t \in C'(w)$ then $Aut(C/C(t), s)$ is transitive on $C(s)$;*
- *$Aut(C(s)/C(t))$ is a continuous homomorphic image of $Aut(W/w, x)$;*
- *the kernel of π is an extension of $Aut(C(s)/C(t))$ by the kernel of π_2.*

Proof. Let $K = Aut(C/W)$. By hypothesis, the natural map $j : K \rightarrow Aut(C(x))$ is an isomorphism. Let $K_1 = j^{-1}(Aut(C(x)/C(w)))$. As K is central in $Aut(C)$, K_1 is a normal subgroup of $Aut(C)$. Let C' consist of the K_1-orbits on C, and π_1, π_2 the maps:

$$\pi_1(c) = K_1 c;\ \pi_2(K_1 c) = \pi(c),$$

and $Aut(C')$ the permutations of C' induced by elements of $Aut(C)$ (by Lemma 1.1 this is closed).

Note that π_1 has automorphism group $Aut(C)$ with kernel K_1, and π_2 has automorphism group isomorphic to $Aut(C)/K_1$ with kernel K/K_1. In particular, both π_1 and π_2 are irreducible and superlinked.

Let $gK_1 \in Aut(C'/C'(w), x)$, where $g \in Aut(C)$. We can choose g so the $gb = b$, for $b \in C(w)$. Thus $g \mid C(x) \in Aut(C(x)/C(w), x) = j(K_1)$. Now let $s \in C'(x)$ and $a \in C(s)$. Then $(gK_1)s = K_1(ga)$. There exists a unique $h \in K$

with $h \mid C(x) = g \mid C(x)$ (by Lemma 1.3), and by the above, $h \in K_1$. Thus $gK_1s = K_1ha = s$. So $Aut(C'(x)/C'(w)) = 1$, as required.

Now let $b, b' \in C(s)$, where $s \in C'(x)$. Then there exists $g \in K_1$ with $gb = b'$. But then there exists $h \in Aut(C(x)/C(w), x)$ with $jg = h$. So $ha = a$ for $a \in C(w)$, and $hb = b'$. Thus $h \in Aut(C(s)/C(t))$. So $Aut(C(s)/C(t))$ is indeed transitive on $C(s)$.

To finish off the proof, note that by Lemma 1.4 restriction is an isomorphism from $Aut(C/C(w))$ to $Aut(W/w)$ and this sends $Aut(C/x, C(w))$ to $Aut(W/x, w)$. As $Aut(C(s)/C(t))$ is isomorphic to K_1 and (via j) this is isomorphic to $Aut(C(x)/C(w))$, the rest of the lemma follows. $\Box$

The following provides a better answer to the question at the end of [4] than Corollary 1.18 did, and is a straightforward consequence of that Corollary together with the above lemma.

Corollary 3.2 *Let L be a countable, transitive, irreducible $\aleph_0$-categorical structure, and suppose that (P, Q, R) is a graphic triple of types for L. Suppose that there exists $r' \in \mathbf{N}$ such that whenever $Q(w, x)$ holds, the rank of a finite abelian homomorphic image of $Aut(W/w, x)$ is at most r' (—this holds in particular if $Aut^0(W/w, x)$ is of finite index in $Aut(W/w, x)$). Then there exists an integer r such that the rank of the kernel of any untwisted, irreducible superlinked finite cover of L is at most r.* $\Box$

3.2 Locally transitive covers

Suppose, as usual, that W is a transitive, irreducible permutation structure. Let $\pi : M \longrightarrow W$ be an untwisted, transitive, superlinked finite cover of W whose kernel is central in $Aut(M)$. Suppose there is a 2-type R on W such that if $R(w, x)$, then $Aut(M/M(w))$ is transitive on $M(x)$. Let $K = Aut(M/W)$. Now, K is isomorphic to $Aut(M(x))$ by hypothesis, and by Lemma 1.3, the latter acts regularly on $M(x)$. So we get a homomorphism from $Aut(M/M(w), x)$ to K by comparing actions on $M(x)$. By assumption of local transitivity, this is a surjection. Furthermore, restriction (to W) is an isomorphism between $Aut(M/M(w), x)$ and $Aut(W/w, x)$ (as in the proof of Lemma 3.1). Thus we have surjective homomorphisms

$$\phi_{w,x} : Aut(W/w, x) \longrightarrow K$$

and if $g \in Aut(W)$ and $h \in Aut(W/w, x)$ then

$$\phi_{gw,gx}(ghg^{-1}) = \phi_{w,x}(h).$$

Terminology Suppose G is a group acting on a set X. The *kernel* of the action is $\{g \in G : gx = x \,\forall x \in X\}$. We say that G is acting *freely* if for any $x, y \in X$ and $g \in G$, $gx = x$ iff $gy = y$. If, additionally, G is transitive on X, we say that G is acting *regularly*.

With this notation and terminology, we have

Lemma 3.3 *(1) There exists a set T on which $Aut(W)$ acts such that for $w, x \in W$ with $R(w,x)$ there is a finite subset $T(w,x)$ of T on which $Aut(W/w,x)$ acts regularly with the kernel of this action being $ker\phi_{w,x}$, and such that $Aut(M/M(w),x)$ acts freely on $M(x) \cup T(w,x)$. If W arises from a countable $\aleph_0$-categorical first-order structure, the T can be taken as a sort in W^{eq}.*

(2) Suppose $R(w,x), R(w,y)$ and $R(x,y)$. Then for $g \in Aut(W/w,x,y)$ we have

$$\phi_{w,y}(g) = \phi_{w,x}(g)\phi_{x,y}(g).$$

(3) Suppose R is symmetric. Then, for $(w,x) \in R$, we have $\phi_{w,x} = \phi_{x,w}^{-1}$

Proof. (1) Let $N(w,x) = ker\phi_{w,x}$, for $(w,x) \in R$. Choose some $(w_0,x_0) \in R$ and let T consist of the set of cosets of $N(w_0,x_0)$ in $Aut(W)$. Let $T(w_0,x_0)$ be those cosets contained in $Aut(W/w_0,x_0)$, and for $g \in Aut(W)$ let $T(gw_0,gx_0)$ be $gT(w_0,x_0)$. This is well-defined; the stabiliser of $t \in T(w,x)$ is $N(w,x)$; and $Aut(W/w,x)$ acts regularly on $T(w,x)$ with kernel $N(w,x)$ (for $(w,x) \in R$). Moreover, if $g \in Aut(M/M(w),x)$ fixes an element of $M(x)$, then $g|W$ is in $ker\phi_{w,x}$ (by definition of $\phi_{w,x}$), and so fixes each element of $T(w,x)$. The converse is also true. So we get that $Aut(M/M(w),x)$ acts freely on $M(x) \cup T(w,x)$.

Finally, if W is $\aleph_0$-categorical and countable, then as $N(w_0,x_0)$ is an open subgroup of $Aut(W)$ we can regard T as a sort in W^{eq}, by the Ryll-Nardzewski theorem.

(2) Let $g \in Aut(W/w,x,y)$. Let $\hat{g} \in Aut(M/M(w),x,y)$ be the unique element extending g. Suppose g induces α_1, α_2, α_3 on $T(w,x)$, $T(w,y)$, $T(x,y)$ respectively. Then $\hat{g}$ induces α_1 on $M(x)$. So $\alpha_1^{-1}\hat{g}$ fixes $M(x)$ and induces α_3 on $T(x,y)$, therefore it also induces α_3 on $M(y)$. So $\hat{g}$ induces $\alpha_1\alpha_3$ on $M(y)$, and this gives the statement.

(3) Let $g \in Aut(W/w,x)$. Let $\hat{g}$ be the unique element of $Aut(M/M(w))$ which extends g. Let $\alpha \in Aut(M/W)$ be the unique element of K such that $\alpha|M(x) = \hat{g}|M(x)$. Then $\phi_{w,x}(g) = \alpha$. Also, $\alpha^{-1}\hat{g} \in Aut(M/M(x))$ extends g, and so by definition $\phi_{x,w}(g) = \alpha^{-1}$. □

The following, somewhat clumsy, terminology will be useful for the ensuing discussion.

Definition 3.4 Suppose K is a finite abelian group. A family $(\phi_{w,x} : (w,x) \in R)$ of continuous, surjective homomorphisms

$$\phi_{w,x} : Aut(W/w,x) \longrightarrow K$$

is called a *conjugate system of homomorphisms* for (W,R,K) if

- for $g \in Aut(W)$ and $h \in Aut(W/w, x)$ we have

$$\phi_{gw,gx}(ghg^{-1}) = \phi_{w,x}(h);$$

- if $R(w,x)$, $R(w,y)$ and $R(x,y)$ then for $g \in Aut(W/w,x,y)$ we have

$$\phi_{w,y}(g) = \phi_{w,x}(g)\phi_{x,y}(g).$$

If these homomorphisms arise from a finite cover π, as above, we refer to them as the *conjugate family associated to* (π, R).

3.3 Superlinked covers of structures with strong types

In this subsection the permutation structure W arises from a first-order structure and its automorphism group. We shall suppose that W is an irreducible, countable, transitive, $\aleph_0$-categorical structure and that there exists a strong type p over W. We shall describe all the untwisted, transitive, irreducible superlinked finite covers of W. Let (P, R, R) be the graphic triple arising from p.

Let $\pi : M \longrightarrow W$ be an untwisted, transitive, irreducible superlinked finite cover of W. By Corollary 2.4, if $Aut(M(x)/M(w)) = 1$ for $R(w,x)$, then π is trivial. So in the factorisation produced by Lemma 3.1, the cover π_2 is trivial, whence $Aut(M/M(w))$ is transitive on $M(x)$. So the conclusion of Lemma 3.3 holds. We shall now show that (with the above hypotheses on W) there is a converse to Lemma 3.3: given a conjugate system of homomorphisms for (W, R, K) we can construct an untwisted, locally transitive superlinked finite cover with kernel K having this as its conjugate system (Theorem 3.5). There is a unique such cover (Theorem 3.7). We introduce a non-degeneracy condition for conjugate systems, and show that the cover constructed by Theorem 3.5 is irreducible precisely when this condition is satisfied (Theorem 3.8).

Let $G = Aut(W)$ and suppose that for some $(w,x) \in R$ we have a closed normal subgroup $N(w,x)$ of $Aut(W/w,x)$ such that the quotient group is a finite abelian group A. Then, by conjugation in $Aut(W)$, the same is true for every $(w,x) \in R$, and so we have surjective homomorphisms

$$\phi_{w,x} : Aut(W/w,x) \longrightarrow A$$

with closed kernels, and for any $g \in Aut(W)$ and $h \in Aut(W/w,x)$

$$\phi_{gw,gx}(ghg^{-1}) = \phi_{w,x}(h).$$

If these homomorphisms satisy the conclusion of Lemma 3.3(2), then we can construct a locally transitive (with respect to (w,x)) superlinked finite cover of W with these forming its conjugate system. However, this cover will not be irreducible unless we assume a further non-degeneracy condition (this condition is both necessary and sufficient, given the other hypotheses). Say that $\phi_{w,x}$ as above is *degenerate* if, for $w' \in W$, there exist homomorphisms

$$\phi_{w'} : Aut(W/w') \longrightarrow A$$

with closed kernels, such that for any $g \in Aut(W)$ and $h \in Aut(W/w')$

$$\phi_{w'}(g) = \phi_{gw}(ghg^{-1})$$

and for any $k \in Aut(W/w, x)$, where $R(w, x)$, we have

$$\phi_{w,x}(k) = \phi_w(k)\phi_x(k)^{-1}.$$

We say that $\phi_{w,x}$ is *fully non-degenerate* if for any non-trivial homomorphism ψ with domain A, $\psi\phi_{w,x}$ is non-degenerate. (Note that if $\phi_{w,x}$ is degenerate, then the condition in 3.3(2) holds.)

Theorem 3.5 *Suppose that W is a countable, transitive $\aleph_0$-categorical structure with a strong type p and associated graphic triple of types (P, R, R). Let A be a finite abelian group and $\Phi = (\phi_{w,x} : (w, x) \in R)$ a conjugate system of homomorphisms for (W, R, A). Then there is an untwisted, transitive, locally transitive superlinked finite cover $\pi : M \longrightarrow W$ with kernel isomorphic to A, having Φ as its associated conjugate system. If Φ is fully non-degenerate, then π is irreducible.*

Proof. As in the proof of (1) of Lemma 3.3, there exists a sort $T \subseteq W^{eq}$ and for every $(w, x) \in R$ a finite subset $T(w, x)$ of T on which $Aut(W/w, x)$ acts regularly with kernel $N(w, x)$. Thus, $\phi_{w,x}$ gives a faithful, regular action of A on $T(w, x)$. Suppose now that we take

- for every $x \in W$ a finite set $M(x)$ on which A acts faithfully and regularly, and $a(x)$ a distinguished element of $M(x)$;
- a function $t : R \longrightarrow T$ such that for each $(w, x) \in R$ we have $t(w, x) \in T(w, x)$.

We define a three-sorted structure (M, W, T) with domain $M \sqcup W \sqcup T$, where $M = \bigsqcup_{x \in W} M(x)$, and with relations

- the existing structure on $W \sqcup T$;
- the projection map $\pi : M \longrightarrow W$ with $\pi^{-1}(x) = M(x)$;
- for each $\alpha \in A$ the set $\{(a, \alpha a) : a \in M\}$;
- a new ternary relation $S \subseteq M \times M \times T$ given by

$$\{(\alpha_1 a(w), \alpha_2 a(x), \alpha_3 t(w, x)) : R(w, x), \alpha_i \in A, \alpha_2 = \alpha_1\alpha_3\}.$$

We shall abuse notation (considerably) and denote this structure by M (the other sorts can be regarded as part of M^{eq}). We shall investigate what conditions on t guarantee that π is a finite cover, and then show that our hypotheses allow us to find such a t. First, however, we show

Claim 1 $Aut(M/W) \cong A$.

Let $x_0 \in W$. Then $Aut(M/W, a(x_0)) = 1$. Indeed, suppose $R(w, x_0)$. Then there is a unique element $a \in M(w)$ with $S(a, a(x_0), t(w, x_0))$, namely $a(w)$. So $Aut(M(w)/W, a(x_0)) = 1$ and the assertion now follows from connectedness of R. Now, for $\alpha \in A$ define $g_\alpha : M \longrightarrow M$ by $g_\alpha a = \alpha a$, for $a \in M$. Then g_α fixes each fibre of π and preserves S, so is in $Aut(M/W)$. This proves the claim.

Let $w \in W$ and $g \in Aut(W)$. If M really is a finite cover of W then g extends to some $\hat{g} \in Aut(M)$ with $\hat{g}a(w) = a(gw)$. This $\hat{g}$ is unique, and it can be obtained by first extending g to a map $M(x) \rightarrow M(gx)$ where $R(w, x)$, then to vertices at distance 2 from w, etc. At each stage, the extension is uniquely determined, and so extending along different paths from w to x should give the same map from $M(x)$ to $M(gx)$. As any two paths between x and w are homotopic it will be enough to ensure that the extension is the same along paths which are elementarily homotopic. To investigate this we introduce the following notation:

Notation Define the map

$$D_t : G \longrightarrow A^R$$

by setting, for $g \in G$ and $(x, y) \in R$,

$$(D_t g)(x, y) = l$$

where l is the unique element of A such that $gt(x, y) = lt(gx, gy)$.

The following perhaps explains the choice of notation (cf. the chain rule for differentiation)

Claim 2 For $g, h \in G$ and $(x, y) \in R$ we have

$$D_t(hg)(x, y) = (D_t g)(x, y)(D_t h)(gx, gy).$$

To see this, just note that

$$hgt(x, y) = h(D_t g(x, y)t(gx, gy)) = (D_t g)(x, y)ht(gx, gy)$$

$$= (D_t g)(x, y)(D_t h)(gx, gy)t(hgx, hgy).$$

Again, fix $w \in W$ and $g \in Aut(W)$. For paths p in the digraph $(W; R)$ starting at w we define a partial map g_p by induction on the length of the path as follows

- g_w is g together with the map $M(w) \longrightarrow M(gw)$ taking $\alpha a(w)$ to $\alpha a(gw)$, for $\alpha \in A$;
- if $g_{w\ldots x}$ has been defined, and $R(x, y)$, then let $g_{w\ldots xy}$ be the union of $g_{w\ldots x}$ together with the function $M(y) \to M(gy)$ which takes $\alpha a(y)$ to $\alpha\alpha_1(D_t g(x, y))a(gy)$, where α_1 is the unique element of A with $g_{w\ldots x}a(x) = \alpha_1 a(gx)$;
- if $g_{w\ldots x}$ has been defined, and $R(y, x)$, then let $g_{w\ldots xy}$ be the union of $g_{w\ldots x}$ together with the function $M(y) \to M(gy)$ which takes $\alpha a(y)$ to $\alpha\alpha_2(D_t g(y, x))^{-1}a(gy)$, where α_2 is the unique element of A with $g_{w\ldots x}a(x) = \alpha_2 a(gx)$.

We now work out what properties t must have for the union of the g_p's to be a function. (It is clear that if it is a function, then it is an automorphism of M extending g.)
Suppose that R is symmetric, that $R(x, y)$, and p is a path from w to x. Then according to the above, we have both

$$g_{py}(a(y)) = \beta(D_t g(x, y))^{-1}a(gy)$$

and

$$g_{py}(a(y)) = \beta(D_t g(y, x))a(gy)$$

where $g_p a(x) = \beta a(gx)$. So for consistency, we require

$$D_t g(x, y) = ((D_t g(y, x))^{-1}. \tag{1}$$

We now consider what consistency conditions are imposed by elementary homotopies (whether R is symmetric or not). First, suppose that $R(x, y)$ and p is a path from w to x. Then

$$g_p a(x) = g_{pyx}a(x)$$

is always true if R is anti-symmetric, and if R is symmetric it is true precisely when equation 1 holds. The same is also true if $R(y, x)$.

So now consider the other type of elementary homotopy. Suppose $R(x, y)$, $R(y, z)$ and $R(x, z)$ and p is a path from w to x. We require

$$g_{pyz}a(z) = g_{pz}a(z)$$

and this happens if and only if

$$(D_t g)(x, y)(D_t g)(y, z) = (D_t g)(x, z). \tag{2}$$

The above discussion now implies:

Claim 3 An automorphism g of W can be extended to an automorphism of M sending $a(w)$ to $a(gw)$ if and only if the equations 1 and 2 hold.

We now show that it is possible to choose t so that equations 1 and 2 do indeed hold. Again, fix $w \in W$. Remember that $w^+ = \{x \in W : R(w,x)\}$. For $x \in w^+$, choose $t(w,x) \in T(w,x)$ arbitrarily. Now, $Aut(W/w)$ is transitive on $R \cap (w^+)^2$. Choose a representative (x_0, y_0), and pick $t(x_0,y_0) \in T(x_0,y_0)$ arbitrarily. For $g \in Aut(W/w)$ let

$$t(gx_0, gy_0) = D_t g(w,x_0)(D_t g(w,y_0))^{-1} g(t(x_0,y_0)).$$

Claim 4 This is well-defined.

Let $g' \in Aut(W/w, x_0, y_0)$. We must show that $t(gx_0, gy_0)$ and $t(gg'x_0, gg'y_0)$ as given by the above recipe are in fact the same. But using the equation in Claim 2, the second of these is the first, multiplied by

$$D_t g'(w,x_0)(D_t g'(w,y_0))^{-1} D_t g'(x_0,y_0)$$

and by hypothesis, this is the identity.

Claim 5 For $x, y \in w^+$ with $R(x,y)$ and $g \in Aut(W/w)$ we have

$$(D_t g)(x,y) = (D_t g)(w,x)((D_t g)(w,y))^{-1}.$$

To see this, write $(x,y) = h(x_0,y_0)$ for some $h \in Aut(W/w)$. So

$$g(t(x,y)) = g(D_t h(w,x_0)(D_t h(w,y_0))^{-1} h(t(x_0,y_0)))$$

$$= D_t h(w,x_0)(D_t h(w,y_0))^{-1} gh(t(x_0,y_0)).$$

Also, we can compute $t(gx,gy) = t(ghx_0, ghy_0)$ using the recipe, and comparing these two gives the required value for $D_t g(x,y)$.

Claim 6 Equations 1 and 2 hold for $g \in Aut(W/w)$ and $x,y,z \in w^+$.

Now we make use of the following (where we are identifying the group-theoretic concept of a type with the usual model-theoretic meaning of the word)

Lemma 3.6 *Suppose W is a countable saturated first-order structure, p a strong type over W, and H a countable subgroup of $Aut(W)$. Then there exists an elementary extension V of W which is isomorphic to W, and $w \in V \setminus W$ such that*

(i) $tp(w/X) = p \mid X$ for all finite $X \subseteq W$;

(ii) $Aut(V/w) \mid W \geq H$.

Proof of lemma. Let C be a strongly ω_1-homogeneous elementary extension of W (see [8], Exercise 10.1.4 for this). Then C is also ω_1-saturated, and so there exists $w \in C$ satisfying (i).

We now show that any automorphism of W extends to an automorphism of C which fixes w. First, note that

$$\bigcup_{X \subseteq W,\, finite} p \mid X$$

is a complete type over W. Now let $h \in Aut(W)$. By ω_1-homogeneity this extends to $\hat{h} \in Aut(C)$. Let $\overline{W}$ be an enumeration of W. Then $tp(w, \overline{W}) = tp(\hat{h}w, \overline{hW})$. Thus

$$tp(\hat{h}w/W) = \bigcup_{X \subseteq W,\, finite} p \mid hX = tp(w/W).$$

Thus there exists $k \in Aut(C/W)$ with $k\hat{h}w = w$. Then $k\hat{h}$ extends h and fixes w.

For every $h \in H$ let $\hat{h} \in Aut(C/w)$ extend h, and let $\hat{H}$ be the subgroup of $Aut(C)$ which these generate. Note that this is countable. There exists a chain of countable subsets of C:

$$W_0 = W \subseteq V_1 \subseteq W_1 \subseteq V_2 \ldots$$

such that

- V_i is a countable, saturated, elementary submodel of C which contains W_{i-1} and w;
- W_i is countable and invariant under $\hat{H}$, and contains V_i.

Then $V = \cup_i V_i$ is a countable, saturated elementary submodel of C (so isomorphic to W) which is $\hat{H}$-invariant. $\Box$ Lemma

Now we return to the proof of the Theorem. Let H be a dense subgroup of $Aut(W)$. Find $V \succeq W$ and $w \in V \setminus W$ as in the lemma. By Claim 6 (working in V rather than W) there exists a t such that equations 1 and 2 hold for $x, y, z \in W$ and $g \in H$. Now use *this* t (with its arguments restricted to W) to construct M. Then by Claim 3, any automorphism in H extends to an automorphism of M. But by Lemma 1.1 the restriction map $Aut(M) \longrightarrow Aut(W)$ has closed image. As H is dense in $Aut(W)$ this means that restriction is surjective, and so $\tau : M \longrightarrow W$ is a finite cover. The kernel of τ is isomorphic to A (by claim 1), and it is clear that the same is true of the fibre group. So τ is untwisted.

To finish the proof of the Theorem, it remains to show that if Φ is fully non-degenerate, then the cover we have constructed is irreducible. It will be enough (by taking a quotient over a central subgroup if necessary) to show that it is non-split. Suppose to the contrary that $\Sigma \leq \Gamma$ is a closed complement to $Aut(M/W)$. If $g \in Aut(W/w)$ there is a unique $\hat{g} \in \Sigma$ which restricts to g. Define $\phi_w(g)$ to be the unique element of A such that $\hat{g}a(w) = \phi_w(g)a(w)$. Then $\phi_w : Aut(W/w) \longrightarrow A$ is a continuous homomorphism, and for $h \in Aut(W)$ we have $\phi_w(g) = \phi_{hw}(hgh^{-1})$. Now let $g \in Aut(W/w, x)$. Then

$$\hat{g}a(w) = \phi_w(g)a(w);\ \hat{g}a(x) = \phi_x(g)a(x)$$

and

$$\hat{g}t(w,x) = gt(w,x) = \phi_{w,x}(g)t(w,x).$$

But $\hat{g}$ must preserve the relation S on M and so

$$\phi_x(g) = \phi_w(g)\phi_{w,x}(g).$$

This contradicts non-degeneracy of $\phi_{w,x}$. □

Theorem 3.7 *Let W be a transitive, irreducible permutation structure with a strong type p and associated graphic triple of types (P, R, R). Suppose $\pi : M \longrightarrow W$ and $\pi' : M' \longrightarrow W$ are untwisted, R-locally transitive superlinked finite covers whose kernels are central in their automorphism groups. Let $(\phi_{w,x} : (w,x) \in R)$ and $(\phi'_{w,x} : (w,x) \in R)$ be the associated conjugate systems of homomorphisms for (π, R) and (π', R) respectively. Suppose there is an isomorphism $\alpha : Aut(M/W) \longrightarrow Aut(M'/W)$ such that $\phi'_{w,x} = \alpha\phi_{w,x}$ for all $(w,x) \in R$. Then there is a bijection $\mu : M \longrightarrow M'$ and an isomorphism $\lambda : Aut(M) \longrightarrow Aut(M')$ such that*

$$\mu(\pi^{-1}(w)) = \pi'^{-1}(w)\ \forall w \in W$$

and

$$(\mu, \lambda) : \langle M; Aut(M)\rangle \longrightarrow \langle M'; Aut(M')\rangle$$

is an isomorphism of permutation structures.

Proof. The first step in the proof is to show that each of π, π' can be constructed as in Theorem 3.5. Let T be as in Lemma 3.3 (1). Note that as $ker\phi_{w,x} = ker\phi'_{w,x}$ for $(w,x) \in R$ this is the same set for each of π and π'. Choose functions $a : W \longrightarrow M$ and $a' : W \longrightarrow M'$ so that $a(w) \in M(w)$ and $a'(w) \in M'(w)$ for all $w \in W$. Let (w_0, x_0) be an arbitrary element of R, and $t(w_0, x_0) \in T(w_0, x_0)$. For any $(w,x) \in R$ there exists $g_1 \in Aut(M)$ with $g_1(a(w_0), a(x_0)) = (a(w), a(x))$ (by the transitivity assumptions), and furthermore if $g_2 \in Aut(M)$ also has this property, then $g_2^{-1}g_1 \in Aut(M/M(w_0), M(x_0))$ and so $g_1(t(w_0,x_0)) = g_2(t(w_0,x_0))$. Thus we define $t(w,x)$ to be $g_1t(w_0,x_0)$. This gives a function $t : R \longrightarrow T$. Define $t' : R \longrightarrow W$ similarly, using π'.

Claim 1 If $g \in Aut(W)$ and $(w, x) \in R$ then

$$gt(w, x) = \phi_{w,x}(g_1^{-1} g) t(gw, gx)$$

where $g_1 \in Aut(M)$ is such that $g_1(a(w)) = a(gw)$ and $g_1(a(x)) = a(gx)$.

First, we show this in the case where $(w, x) = (w_0, x_0)$. So

$$gt(w_0, x_0) = g_1(g_1^{-1} g) t(w_0, x_0) = \phi_{w_0,x_0}(g_1^{-1} g) g_1 t(w_0, x_0),$$

which is what we want.

Now let $h \in Aut(M)$ be such that $h(a(w_0)) = a(w)$ and $h(a(x_0)) = a(x)$. Then

$$gt(w, x) = ght(w_0, x_0) = \phi_{w_0,x_0}((g_1 h)^{-1}(gh)) t(ghw_0, ghx_0)$$
$$= \phi_{hw_0,hx_0}(g_1^{-1} g) t(ghw_0, ghx_0)$$

as required. ($\square$ Claim 1)

Claim 2 The function t satisfies the equations (1) and (2) in Claim 2 of the proof of Theorem 3.5: if $P(x, y, z)$ and $g \in Aut(W)$ then:

$$D_t g(x, z) = D_t g(x, y) D_t g(y, z);$$

and if R is symmetric then

$$D_t g(x, y) = (D_t g(y, x))^{-1}.$$

(Of course, a similar statement holds for t'.)

First, suppose that R is symmetric. Let $g_1 \in Aut(M)$ be such that $g_1 a(x) = a(gx)$ and $g_1 a(y) = a(gy)$. Then by claim 1, $D_t g(x, y) = \phi_{x,y}(g_1^{-1} g)$ and $D_t g(y, x) = \phi_{y,x}(g_1^{-1} g)$. But $\phi_{x,y} = \phi_{y,x}^{-1}$. This gives equation (1).

For equation (2), note that by Lemma 2.2 there exist $k, l, m \in Aut(M)$ and $\Omega \in Aut(M/W)$ such that

$$ka(x) = a(gx), ka(y) = a(gy);$$

$$la(x) = \Omega a(gx), la(y) = a(gy), la(z) = a(gz);$$

$$ma(x) = a(gx), ma(z) = a(gz).$$

So in particular, $l^{-1} g \in Aut(W/x, y, z)$, $k^{-1} g \in Aut(M/x, y)$ and $m^{-1} g \in Aut(W/x, z)$.

Now,

$$D_t g(x, y) D_t g(y, z) = \phi_{x,y}(k^{-1} g) \phi_{y,z}(l^{-1} g)$$
$$= \phi_{x,y}(k^{-1} g (l^{-1} g)^{-1}) \phi_{x,y}(l^{-1} g) \phi_{y,z}(l^{-1} g)$$
$$= \phi_{x,y}(k^{-1} l) \phi_{x,z}(l^{-1} g).$$

Also, $k^{-1}l(a(x)) = \Omega a(x)$; $k^{-1}l(a(y)) = a(y)$; $m^{-1}l(a(x)) = \Omega a(x)$ and $m^{-1}l(a(z)) = a(z)$. So

$$\phi_{x,y}(k^{-1}l) = \Omega^{-1} = \phi_{x,z}(m^{-1}l).$$

As

$$\phi_{x,z}(m^{-1}g)(\phi_{x,z}(l^{-1}g))^{-1} = \phi_{x,z}(m^{-1}l)$$

we get

$$D_t g(x,z) = \phi_{x,z}(m^{-1}g) = \Omega^{-1}\phi_{x,z}(l^{-1}g)$$
$$= \phi_{x,y}(k^{-1}l)\phi_{x,z}(l^{-1}g) = D_t g(x,y) D_t g(y,z)$$

as required. (I am indebted to Jeffrey Koshan for this argument.) (□ Claim 2)

Now, for $(w,x) \in R$ let $\delta(w,x)$ be the unique element of $A = Aut(M/W)$ such that $t'(w,x) = \delta(w,x)t(w,x)$ (– perhaps it is worth emphasising here that $T(w,x)$ is a set of cosets in $Aut(W)$ of some $ker\phi_{w',x'}$, and A is naturally isomorphic to $Aut(W/w,x)/ker\phi_{w,x}$).

Claim 3 For $(w,x) \in R$ and $g \in Aut(W)$ we have

$$D_{t'} g(w,x) = \alpha(\delta(w,x)\delta(gw,gx)^{-1} D_t g(w,x)).$$

This follows easily from the definitions. (□ Claim 3)

Claim 4 Without loss of generality, we can assume that if $P(x,y,z)$, then

$$\delta(x,y)\delta(y,z) = \delta(x,z).$$

Moreover, if R is symmetric then $\delta(y,z) = \delta(z,y)^{-1}$.

It follows from Claim 3 and the fact that Equation 2 holds for both t and t', that for any $g \in Aut(W)$ we have

$$\delta(x,y)\delta(gx,gy)^{-1}\delta(y,z)\delta(gx,gy)^{-1} = \delta(x,z)\delta(gx,gz)^{-1}.$$

From this, and the fact that $Aut(W)$ is transitive on P, it follows that

$$\delta(x,y)\delta(y,z)\delta(x,z)^{-1}$$

is independent of the choice of $(x,y,z) \in P$. Thus, if instead of t' we consider a function which is a mutiple of it by some suitable element of $Aut(M'/W)$ (in its action on T), we obtain a function for which Equations (1) and (2) still hold (indeed, $D_{t'}$ is unchanged), and for which the claim also holds.

Finally, suppose R is symmetric. Then from what we have just proved,

$$\delta(x,z) = \delta(x,y)\delta(y,z)$$

and

$$\delta(x,y) = \delta(x,z)\delta(z,y).$$

Multiplying these together gives what we want. ($\Box$ Claim 4)

We now define a function $\mu : M \longrightarrow M'$. We fix $w \in W$, we set $\mu(a(w)) = a'(w)$ and define $\mu(a(x))$ by induction on the distance of x from w (in the sense of R). Once this is done, we let $\mu(\beta a(x)) = \alpha(\beta)\mu(a(x))$ for $\beta \in A$. So suppose $wx_1 \dots x_r$ is a path. Suppose $\mu(a(x_{r-1}))$ has been defined and equals $\Omega a'(x_{r-1})$ (for $\Omega \in Aut(M'/W)$). Then we put

$$\mu(a(x_r)) = \Omega\alpha(\delta(x_{r-1}, x_r)^{-1})a'(x_r)$$

if $R(x_{r-1}, x_r)$, and

$$\mu(a(x_r)) = \Omega\alpha(\delta(x_{r-1}, x_r))a'(x_r)$$

if $R(x_r, x_{r-1})$.

Claim 5 The function μ is a well-defined bijection.

The non-obvious point here is that μ is well-defined. This follows from Claim 4, using the fact that any two paths with the same endpoints are elementarily homotopic (as in the proof of Theorem 3.5). ($\Box$ Claim 5)

Finally, we define an isomorphism $\lambda : Aut(M) \longrightarrow Aut(M')$ which commutes with μ. Let $w \in W$ be as in Claim 5. Let $g_1 \in Aut(M)$ be such that $g_1 a(w) = a(g_1 w)$. Let $w = y_0 y_1 \dots y_s = g_1 w$ be a path in the digraph $(W; R)$ from w to $g_1 w$. Let $\zeta_i = +1$ if $R(y_i, y_{i+1})$ and -1 otherwise. Then there is a unique $\lambda(g_1) \in Aut(M')$ such that

$$\lambda(g_1)|W = g_1|W$$

and

$$\lambda(g_1)a'(w) = \alpha(\delta(y_0, y_1)^{\zeta_0} \dots \delta(y_{s-1}, y_s)^{\zeta_{s-1}})^{-1}a'(g_1 w).$$

We also define, for $\beta \in Aut(M/W)$

$$\lambda(\beta g_1) = \alpha(\beta)\lambda(g_1).$$

Claim 6 With the above notation, for all $x \in W$,

$$\lambda(g_1)\mu(a(x)) = \mu(g_1 a(x)).$$

Let $x_0 x_1 \dots x_r$ be a path in W (with respect to R) with $w = x_0$ and $x = x_r$. Let η_i equal 1 if $R(x_i, x_{i+1})$ and -1 otherwise. Then, by definition,

$$\mu(a(x)) = \alpha(\delta(x_0, x_1)^{\eta_0} \dots \delta(x_{r-1}, x_r)^{\eta_{r-1}})^{-1}a'(x)$$

and similarly

$$\mu(a(gx)) = \alpha(\delta(y_0, y_1)^{\zeta_0} \dots \delta(y_{s-1}, y_s)^{\zeta_{s-1}}\delta(gx_0, gx_1)^{\eta_0} \dots \delta(gx_{r-1}, gx_r)^{\eta_{r-1}})^{-1}a'(gx)$$

where $g = g_1|W$.

Now

$$g_1 a(x) = (D_t g(x_0, x_1)^{\eta_0} \ldots D_t g(x_{r-1}, x_r)^{\eta_{r-1}}) a(gx)$$

and

$$\begin{aligned}&\lambda(g_1)\mu(a(x)) = \\ &\quad \alpha(\delta(x_0, x_1)^{\eta_0} \ldots \delta(x_{r-1}, x_r)^{\eta_{r-1}})^{-1} \\ &\quad \alpha(\delta(y_0, y_1)^{\zeta_0} \ldots \delta(y_{s-1}, y_s)^{\zeta_{s-1}})^{-1} \\ &\quad D_{t'} g(x_0, x_1)^{\eta_0} \ldots D_{t'} g(x_{r-1}, x_r)^{\eta_{r-1}} a'(gx).\end{aligned}$$

The desired equality now follows from Claim 3. (□ Claim 6)

So λ commutes with μ. All that remains to be checked, is that λ is actually a homomorphism. Let $f, g \in Aut(M)$. We can assume that $ga(w) = a(gw)$ and $fa(w) = a(fw)$. As λ followed by restriction to W is clearly a homomorphism, it is only necessary to show that

$$\lambda(fg)a'(w) = \lambda(f)\lambda(g)a'(w).$$

We leave the details of the proof of this to the reader. □

Theorem 3.8 *Let W be a transitive, irreducible permutation structure with a strong type p and associated graphic triple of types (P, R, R). Suppose $\pi : M \longrightarrow W$ is an irreducible, untwisted, R-locally transitive superlinked finite cover. Then $(\phi_{w,x} : (w,x) \in R)$, the associated conjugate system of homomorphisms for (π, R), is fully non-degenerate.*

Proof. Let $K = Aut(M/W)$. It is enough to show that if $\phi_{w,x} = \phi_w \phi_x^{-1}$, for some conjugate system of homomorphisms

$$\phi_w : Aut(W/w) \longrightarrow K$$

then the cover π is split. We do this by constructing a *split*, untwisted, locally transitive superlinked finite cover $\pi' : M' \longrightarrow W$ whose kernel is central in $Aut(M')$, which has $(\phi_w \phi_x^{-1} \colon (w,x) \in R)$ as its associated conjugate system of homomorphisms. The result will then follow from Theorem 3.7.

First, we form a regular, trivial cover $\pi_0 : M_0 \longrightarrow W$ with fibre groups $\phi_w(Aut(W/w))$. So we have an action of K on each fibre of π_0. Then we take $\pi' : M' \longrightarrow W$ to be a reduct of π_0, obtained by introducing further automorphisms into the kernel. For $k \in K$ and $w \in W$ we let $\hat{k}|M'(w) = \phi_w^{-1}(k)|M_0(w)$. So $Aut(M')$ is generated by $Aut(M_0)$ and $\{\hat{k} : k \in K\}$. Let $c \in M'$ and $h \in Aut(M_0)$. Then, for $k \in K$ we have

$$h\hat{k}(c) = h\phi_{\pi' c}^{-1}(k)(c) = (\phi_{\pi' c}^{-1}(k))^h (hc)$$

$$= \phi^{-1}_{\pi'(hc)}(k)(hc) = \hat{k}h(c).$$

Thus $Aut(M')$ is a direct product of $Aut(M_0)$ and $\{\hat{k} : k \in K\}$. It follows that π' is an untwisted, locally transitive superlinked finite cover with kernel isomorphic to K. It remains to compute the associated conjugate system for this cover. Let $h \in Aut(W/w, x)$. Then h gives rise to a unique $h_1 \in Aut(M_0)$, and $(\phi_w(h))^{-1}h_1$ is in $Aut(M')$; extends h, and is trivial on $M'(w)$. Thus

$$\phi_{w,x}(h) = \phi_w(h)^{-1}\phi_x(h)$$

which is enough for our purposes. □

Corollary 3.9 *Let W be a transitive, irreducible countable $\aleph_0$-categorical structure with a strong type p and associated graphic triple of types (P, R, R). Let K be a finite abelian group. Then there is a one-to-one correspondence between untwisted, irreducible superlinked finite covers of W with kernel K and fully non-degenerate conjugate families of homomorphisms for (W, R, K).*

Remark The method of producing finite covers on page 589 of the paper [3] by E. Bouscaren and E. Hrushovski provides an alternative (and simpler) approach to the construction of the covers in the above theorem (that is, it is an alternative approach to Theorem 3.5). It is not clear, however, whether this approach will provide the full classification result.

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